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Nonparametric classes of lifetime distributions via binary associative operation

B. L. S. Prakasa Rao

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

NONPARAMETRIC CLASSES OF LIFETIME DISTRIBUTIONS VIA BINARY ASSOCIATIVE OPERATION

B.L.S.Prakasa Rao
Indian Statistical Institute, New Delhi

Abstract

A binary operation * over real numbers is said to be associative if $(x*y)*z = x*(y*z)$ and it is said to be reducible if $x*y = x*z$ or $y*w = z*w$ if and only if $z = y$. The operation * is said to have an identity element $\tilde{e}$ if $x*\tilde{e} = x$. We study different classes of lifetime probability distributions under binary associative operations between random variables.

1 Introduction

Let T be a positive random variable with distrbution function F which can be interpreted as the lifetime of a component or system. If the distribution function has a probability density f, then $\lambda(t) = \frac{f(t)}{\bar{F}(t)}$ is called the failure rate or hazard rate where $\bar{F}(t) = 1 - F(t)$ is the survival probability. Interpreting T as the lifetime of some component or system, the failure rate measures the proneness to failure at time t. It is well known that

$$\bar{F}(t) = \exp\{-\int_0^t \lambda(s)\ ds\}$$

which shows that F is uniquely determined by the failure rate. Aging of a system is reflected in the properies of its failure rate. If $\lambda(.)$ is increasing, then the distribution F is said to have the IFR (increasing failure rate property). Analogous properies of distributions such as DFR (decreasing failure rate), IFRA (increasing failure rate average), NBU (new better than used) and NBUE(new better than used in expectation) are discussed in the literature to describe the aging of a system. We will now extend these notions using the concept of a binary associative operation. A binary operation * over real numbers is said to be associative if

$$(x*y)*z = x*(y*z) \tag{1. 1}$$

for all real numbers x, y, z. The binary operation * is said to be reducible if $x*y = x*z$ if and only if $y = z$ and if $y*w = z*w$ if and only if $y = z$. It is known that the general reducible continuous solution of the functional equation (1.1) is

$$x*y = g^{-1}(g(x) + g(y)) \tag{1. 2}$$

where $g(.)$ is a continuous and strictly monotone function provided $x, y, x*y$ belong to a fixed (possibly infinite) interval A (cf. Aczel (1966)). The function g in (1.2) is determined up to a multiplicative constant, that is,

$$g_1^{-1}(g_1(x) + g_1(y)) = g_2^{-1}(g_2(x) + g_2(y))$$

for all x, y in a fixed interval A implies $g_2(x) = \alpha g_1(x)$ for all x in that interval for some $\alpha \neq 0$. We assume here after that the binary operation is reducible and associative with the function $g(.)$ continuous and strictly increasing. Further assume that there exists an identity element $\tilde{e} \in \bar{R}$ such that

$$x * \tilde{e} = x, x \in A.$$

It is also known that every continuous, reducible and associative operation defined on an interval A in the real line is commutative (cf. Aczel (1966), p.267). Let X be a random variable with the distribution function $F(x)$ having support A. Define

$$\phi_X^*(s) = \int_A \exp\{isg(x)\} \; F(dx), -\infty < s < \infty. \tag{1. 3}$$

Note that the function $\phi_X^*(s)$ is the characteristic function of the random variable $g(X)$ and hence determines the distribution function of the random variable $g(X)$ uniquely. Examples of such binary operations are given in Castagnoli (1974, 1978, 1982), Muliere (1984) and Castagnoli and Muliere (1984, 1986, 1988). For instance (i) if $A = (-\infty, \infty)$ and $x * y = x + y$, then $g(x) = x$, (ii) if $A = (0, \infty)$ and $x * y = xy, x > 0, y > 0$ then $g(x) = \log x$, (iii) if $A = (0, \infty)$ and $x * y = (x^\alpha + y^\alpha)^{1/\alpha}, x > 0, y > 0$ for some $\alpha > 0$, then $g(x) = x^\alpha$, (iv) if $A = (-1, \infty)$ and $x * y = x + y + xy + 1, x > -1, y > -1$, then $g(x) = \log(1 + x)$ (v)if $A = (0, \infty)$ and $x * y = xy/(x + y), x > 0$, then $g(x) = 1/x$ and (vi)if $A = (0, \infty)$ and $x * y = (x + y)/(1 + xy), x > 0, y > 0$, then $g(x) = arth\, x$. A characterization of the multivariate normal distribution through a binary operation which is associative is given in Prakasa Rao (1974) and in Prakasa Rao (1977) for Gaussian measures on locally compact abelian groups. Muliere and Scarsini (1987) characterize a class of bivariate distributions that generalize the Marshall-Olkin bivariate exponential distribution through a functional equation involving two associative operations. Muliere and Prakasa Rao (2003) studied characterization of some probability distributions via binary associative operation. They characterize distributions with the almost lack of memory property or strong Markov property or with the periodic failure rate under a binary associative operation extending the properties of exponential distribution under addition operation as binary associative operation. A type of bivariate lack of memory property under binary associative operation was investigated in Prakasa Rao (2004).

2 Nonparametric classes of lifetime distibutions

Let * be a binary operation which is reducible and associative over an interval A contained in R and $g(.)$ be the associated function as described above. Without loss of generality, we assume that the function $g(.)$ is continuous and strictly increasing with identity $\tilde{e} \in \bar{R}$ with the property $x * \tilde{e} = x$ for all $x \in A$. Suppose that T is a random variable with distribution function F. Then

$$P(T > t * x | T > t) = \frac{\bar{F}(t * x)}{\bar{F}(t)}$$

for all $t \in A, x \geq \tilde{e}$ such that $t * x \in A$ and $\bar{F}(t) > 0$. **Definition 2.1:** The distribution function F is said to have an IFR* *(increasing failure rate under the binary associative operation *)* property if $\frac{\bar{F}(t*x)}{\bar{F}(t)}$ is nonincreasing in $t \in A$ for all $x \geq \tilde{e}$ such that $t * x \in A$. The distribution function F is said to have an DFR* *(decreasing failure rate under the binary associative operation *)* property if $\frac{\bar{F}(t*x)}{\bar{F}(t)}$ is nondecreasing in $t \in A$ for all $x \geq \tilde{e}$ such that $t * x \in A$.

Let $G(.)$ be the distribution function of the random variable $g(T)$. Then

$$\begin{aligned} P(T > t * g^{-1}y | T > t) &= P(g(T) > g(t * g^{-1}y) | T > t) \\ &= P(g(T) > g(t) + y | T > t) \\ &= P(g(T) > g(t) + y | g(T) > g(t)) \\ &= \exp\{-\int_{g(t)}^{g(t)+y} \lambda_G(s) ds\} \end{aligned}$$

where $\lambda_g(.)$ is the failure rate corresponding to the distribution function of $g(T)$. This follows from the fact that the function $g(.)$ is continuous and strictly increasing. Hence

$$P(T > t * x | T > t) = \exp\{-\int_{g(t)}^{g(t)+g(x)} \lambda_G(s) ds\}.$$

Following the above remarks, we define the IFRA* and DFRA* properties.

Definition 2.2: A distribution function F is said to have the IFRA* *(increasing failure rate average under the binary associative operation *)* property if $-(1/u) \log \bar{G}(u)$ is nondecreasing on the set $\{u : \bar{G}(u) > 0\}$. It is said to have DFRA* *(decreasing failure rate average under the binary associative operation *)* property if $-(1/u) \log \bar{G}(u)$ is nonincreasing on the set $\{(u : \bar{G}(u) > 0\}$.

It is obvious that $G(y) = Fog^{-1}(y)$ and $\bar{G}(y) = \bar{F}og^{-1}(y)$. Further more

$$\lambda_G(y) = \lambda(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)].$$

Suppose that F is IFRA*. Then

$$(\bar{G}(y))^{1/y}$$

is nondecreasing in the range of y which implies that

$$\bar{G}(\alpha t) \geq (\bar{G}(t))^{\alpha}$$

for $0 \leq \alpha \leq 1$ and $t \geq 0$. Alternatively

$$[\bar{F}(g^{-1}(y))]^{1/y}$$

is nondecreasing in the range of y.

Let

$$\mu_G(t) = \frac{1}{\bar{F}(t)} \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{F}(t * x)dx.$$

Note that

$$\begin{aligned} \mu_G(t) &= \frac{1}{\bar{G}(g(t))} \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(t * x))dx \\ &= \frac{1}{\bar{G}(g(t))} \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(t) + g(x))dx \\ &= E[g(T) - g(t) > g(x)|g(T) > g(t)]. \end{aligned}$$

The function $\mu_G(t)$ reduces to mean residual life time of the distribution of T if the binary associative operation is addition.

Definition 2.3: A distibution function F is said to have NBU* *(new better than used under the binary associative operation*)* property if

$$\bar{F}(x * t) \leq \bar{F}(x)\bar{F}(t)$$

for all x and t such that $x \in A, x * t \in A, \tilde{e} < t < g^{-1}(\infty)$. It is said to have NBUE* *(new better than used in expectation under the binary associative operation *)* property if

$$\mu_G(t) \leq \mu_G = \int_0^\infty \bar{G}(y)d[g^{-1}(y)] < \infty.$$

Note that the constant μ_G reduces to the mean residual life time of the distibution function F if the binary associative operation is addition. One can similarly define NWU* and NWUE* properties by reversing the inequality signs in the above definition.

3 Interelations between different classes of lifetime distrbutions

The following theorem gives the inter relation between different classes of lifetime distributions defined above. **Theorem 3.1:** Let T be a random variable with distribution function F. Suppose that $E(g(T)) < \infty$ where $g(.)$ is a function associated with a binary associatiove operation *. Further suppose that the random variable $g(T) \geq 0$ with probability one and has continuous distribution function. Then the following chain relation holds for the distribution function F:

$$\text{F } IFR* \Rightarrow \text{F } IFRA* \Rightarrow \text{F } NBU* \Rightarrow \text{F } NBUE*.$$

Proof: Suppose the distribution function F has the IFR* property. Let G be the distribution of $g(T)$. Since the distribution F is IFR*, it follows that

$$\frac{\bar{F}(t * x)}{\bar{F}(t)}$$

is nonincreasing in t in the appropriate domain. In other words,

$$\frac{\bar{G}(g(t * x))}{\bar{G}(g(t))}$$

is nonincreasing in t in the appropriate domain. Let $\bar{G}(t) = \exp\{-\Lambda_G(t)\}$. Then it follows that, for every $x \geq g^{-1}(\tilde{e})$,

$$\exp\{\Lambda_G(g(t * x)) - \Lambda_G(t)\}$$

is nondecreasing in t in the appropriate range of t. Therefore

$$\exp\{\Lambda_G(g(t) + g(x)) - \Lambda_G(g(t))\}$$

is nondecreasing in t in the appropriate range of t for every $x \geq g^{-1}(\tilde{e})$. Since the function $g(.)$ is continuous and strictli increasing, it follows that

$$\exp\{\Lambda_G(u + v) - \Lambda_G(v)\}$$

is nondecreasing in v in the appropriate range of v for every $u \geq 0$.. Therefore the the function $\Lambda_G(.)$ is convex which implies that

$$\Lambda_G(\alpha u + (1 - \alpha)v) \leq \alpha \ \ \Lambda_G(u) + (1 - \alpha) \ \ \Lambda_G(v), 0 \leq \alpha \leq 1.$$

Let $v \to 0$ from the right. Then we get that $\Lambda_G(0) = 0$ and it follows from the above convexity property that

$$\Lambda_G(\alpha u) \leq \alpha \Lambda_G(u)$$

or equivalently

$$\bar{G}(\alpha u) \geq (\bar{G}(u))^{\alpha}.$$

This inequality proves that the distribution F has the IFRA* property according to the Definition 2.2. Suppose now that F has the IFRA* property. For $u, v \geq 0$, let $a = -(1/u) \log \bar{G}(u)$ and $b = -(1/v) \log \bar{G}(v)$. It follows from the IFRA* property that

$$-(1/(u + v)) \log \bar{G}(u + v) \geq \max\{a, b\}$$

and hence

$$-\log \bar{G}(u + v) \geq \max\{a, b\}(u + v) \geq au + bv = -\log \bar{G}(u) - \log \bar{G}(v)$$

which implies that

$$\bar{G}(u + v) \leq \bar{G}(u)\bar{G}(v).$$

The last statement in turn shows that

$$\bar{F}og^{-1}(u+v) \leq \bar{F}og^{-1}(u)\bar{F}og^{-1}(v)$$

or equivalently

$$\bar{F}(g^{-1}u * g^{-1}(v)) \leq \bar{F}(g^{-1}(u))\bar{F}(g^{-1}(v))$$

for all $u, v \geq 0$. Hence

$$\bar{F}(x * t) \leq \bar{F}(x)\bar{F}(t)$$

for all $x, t \in A$ such that$t \geq g^{-1}(\tilde{e})$ and $x * t \in A$. This proves that the distribution F has the NBU* property. Suppose now that the distribution F has the NBU* property. Then it follows that

$$\int_{\tilde{e}}^{g^{-1}(\infty)} \bar{F}(x * t)dx \leq \bar{F}(t) \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{F}(x)dx$$

which implies that

$$\int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(t) + g(x))dx \leq \bar{G}(g(t)) \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(x))dx,$$

that is

$$\begin{aligned}
\mu_G(t) &= \frac{1}{\bar{G}(g(t))} \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(t * x)) \ dx \\
&\leq \int_{\tilde{e}}^{g^{-1}(\infty)} \bar{G}(g(x)) \ dx \\
&= \int_0^{\infty} \bar{G}(y)d[g^{-1}(y)] \\
&= \mu_G.
\end{aligned}$$

Hence the distribution function F has the NBUE* property.

Remarks: It is known that the none of the above implications can be reversed when binary associative operation is addition. The same comment continues to hold here.

4 Closure theorems

Suppose we consider a system where components are allowed to have an arbitrary but finite number of states or levels. Let the components be numbered from 1 to n. Let x_i represent the state of the component i. Suppose that x_i can be in one out of $M_i + 1$ states

$$x_{i0}, x_{i1}, \ldots, x_{iM_i}, (x_{i0} < x_{i1} < \ldots, x_{iM_i}).$$

The states may represent different levels of performance of the component, from the worst x_{i0} to the best x_{iM_i}. We call the states $x_{i0}, x_{i1}, \ldots, x_{iM_i}$ as the failure states of the i-th component. Let $\Phi(x_1, \ldots, x_n)$ denote the state of the system. The function Φ is called the *structure function* of the system. Suppose that the function Φ is equal to one if the system is in the functioning state and takes the value 0 if the system is in the failure state. A system is said to be *monotone* if its structure function Φ is nondecreasing in each argument and

$$\Phi(x_{10}, x_{20}, \ldots, x_{no}) = 0 \text{ and } \Phi(x_{1M_1}, x_{2M_2}, \ldots, x_{nM_n}) = 1.$$

Let $p_i = P(x_i = 1), 1 \leq i \leq n$ and $h(\mathbf{p}) = P(\Phi(\mathbf{x}) = 1)$ where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{x}$ denotes the state of the system. The function $h(\mathbf{p})$ is called the *system reliability* or *reliability function* of the system. It is known that if the structure function of a system is monotone, then

$$h(\mathbf{p}^\alpha) \geq [h(\mathbf{p})]^\alpha$$

for $0 < \alpha \leq 1$ where $\mathbf{p}^\alpha = (p_1^\alpha, p_2^\alpha, \ldots, p_n^\alpha)$ (for proof, see Aven and Jensen (1999), Lemma 3, p.41.)

We now prove a closure theorem for IFRA* distributions.

Theorem 4.1: If each of the independent components of a monotone structure has an IFRA* life time distribution, then the system itself has an IFRA* distribution. **Proof:** Let F and $F_i, i = 1, 2, \ldots, n$ be the distributions of the lifetimes T of the system and $T_i, i = 1, 2, \ldots, n$ of the components respectively. Let $g(.)$ be the function associated with the binary associative operation $*$. Let G and G_i be the distribution functions of $g(T)$ and $g(T_i)$ respectively for $i = 1, 2, \ldots, n$. Since the distribution F_i has the IFRA* property, it follows that

$$\bar{G}_i(\alpha t) \geq (\bar{G}_i(t))^\alpha \tag{4. 1}$$

for $0 \leq \alpha \leq 1$ and $t \geq 0$. Note that the distribution F is related to the distributions $F_i, i = 1, 2, \ldots, n$ by the reliability function

$$\bar{F}(t) = h(\bar{F}_1(t), \ldots, \bar{F}_n(t)).$$

note that

$$\begin{aligned}
\bar{G}(\alpha t) &= \bar{F}(g^{-1}(\alpha t)) \\
&= h(\bar{F}_1(g^{-1}(\alpha t)), \ldots, \bar{F}_n(g^{-1}(\alpha t))) \\
&= h(\bar{G}_1(\alpha t), \ldots, \bar{G}_n(\alpha t)) \\
&\geq h(\bar{G}_1(t)^\alpha, \ldots, \bar{G}_n(t)^\alpha) \\
&\geq [h(\bar{G}_1(t), \ldots, \bar{G}_n(t))]^\alpha \\
&= [h(\bar{F}_1(g^{-1}(t)), \ldots, \bar{F}_n(g^{-1}(t))]^\alpha \\
&= [\bar{F}(g^{-1}(t))]^\alpha \\
&= [\bar{G}(t)]^\alpha
\end{aligned}$$

by the the equation (4.1) and the monotonicity of the function $h(.)$ for $0 < \alpha \leq 1$. The inequality holds clearly for $\alpha = 0$ since $G(0) = 0$. Hence the distribution F of the system life T has the IFRA* property.

Theorem 4.2: Let T_1 and T_2 be independent random variables such that their distributions have IFR* property under a binary associative operation *. Then the random variable $T_1 * T_2$ has a distribution with IFR* property. **Proof:** Let $g(.)$ be the function associated with the binary associative operation *. Observe that

$$g(T_1 * T_2) = g(T_1) + g(T_2)$$

Further more $g(T_1)$ and $g(T_2)$ are independent random variables and both of them have distributions with IFR property. Applying Theorem 5 in Aven and Jensen (1999), we obtain that the random variable $g(T_1) + g(T_2)$ has an IFR distribution. Hence the random variable $g(T_1 * T_2)$ has an IFR distribution, that is, the random variable $T_1 * T_2$ has an IFR* distribution.

5 Stochastic comparison

Let T be a random variable with an IFRA* distribution under a binary associative operation *. Let $g(.)$ be the associated function as given in Section 2. Note that $g(.)$ is continuous and strictle increasing with $g(\tilde{e}) = 0$. Suppose that $g(T) \geq 0$ with probability one. Let F be the distribution function of T and G be the distribution function of $g(T)$. Define

$$v_G(t) = (-\log \bar{G}(t))/t.$$

Since the distribution function F has the IFRA* property, it follows that the function $v_G(t)$ is nondecreasing. For $0 < p < 1$, define x_p such that $G(x_p) = p$ or equivalently $F(g^{-1}(x_p)) = p$. Let $\alpha = -\frac{1}{x_p} \log(1-p)$. From the nondecreasing property of the function $v_G(.)$ and the fact that $v_G(t) \leq v_G(x_p)$ for all $t < x_p$ and $v_G(t) \geq \alpha$ for $t \geq x_p$, it follows that

$$\bar{G}(t) \geq \exp\{-\alpha t\} \text{ for } 0 \leq t \leq x_p$$

and

$$\bar{G}(t) \leq \exp\{-\alpha t\} \text{ for } t \geq x_p.$$

Noting that $\bar{G}(t) = \bar{F}(g^{-1}(t))$ and observing that the function is continuous and strictly ingreasing, we obtain the following theorem.

Theorem 5.1: Let T be a random variable with distribution function F with the IFRA* property under a binary associative operation * with the associated function $g(.)$. Suppose that $g(\tilde{e}) = 0$ and $g(T) \geq 0$ with probability one. Then

$$\bar{F}(u) \geq \exp\{-\alpha \ g(u)\} \text{ for } \tilde{e} = g^{-1}(0) \leq u \leq g^{-1}(x_p)$$

and

$$\bar{F}(u) \leq \exp\{-\alpha \ g(u)\} \ \text{ for } \ u \geq g^{-1}(x_p)$$

where

$$\alpha = -\frac{1}{x_p} \log(1-p).$$

Let T be a random variable with a continuous IFR* distribution and $g(.)$ be as defined above. Then $g(T)$ is a positive random variable with an IFR distribution, say, G. Let m_G be its mean. Applying Lemma 7 in Aven and Jensen (1999), we get that

$$\bar{G}(t) \geq \exp\{-t/m_G\} \ \text{ for } \ 0 \leq t < \eta_G$$

and

$$\bar{G}(t) \leq \exp\{-t/m_G\} \ \text{ for } \ t \geq \eta_G$$

where

$$\eta_G = \inf\{t \in R^+ : -\frac{1}{t} \log \bar{G}(t) \geq \frac{1}{m_G}\}.$$

These inequalities can be written in the form

$$\bar{F}(u) \geq \exp\{-g(u)/m_G\} \ \text{ for } \ g^{-1}(0) \leq u \leq g^{-1}(\eta_G)$$

and

$$\bar{F}(u) \leq \exp\{-g(u)/m_G\} \ \text{ for } \ u \geq g^{-1}(\eta_G).$$

Theorem 5.2: Let T be a random variable with distribution function F with the IFR property under a binary associative operation * with the associated function $g(.)$. Suppose that $g(\tilde{e}) = 0$ and $g(T) \geq 0$ with probability one. Let G be the distribution function of $g(T)$. Further suppose that $m_G = E(g(T)) < \infty$. Define

$$\eta_G = \inf\{t \in R^+ : -\frac{1}{t} \log \bar{G}(t) \geq \frac{1}{m_G}\}.$$

Then the following inequalities hold:

$$\bar{F}(u) \geq \exp\{-g(u)/m_G\} \ \text{ for } \ g^{-1}(0) \leq u \leq g^{-1}(\eta_G)$$

and

$$\bar{F}(u) \leq \exp\{-g(u)/m_G\} \ \text{ for } \ u \geq g^{-1}(\eta_G).$$

6 Examples:

(i)**Weibull distribution:** Suppose the random variable T follows a Weibull distribution

$$\bar{F}(u) = \exp\{-u^{\beta}\}, 0 < u < \infty, \beta > 1.$$

Let * be the binary associative operation defined by

$$x * y = (x^{\beta} + y^{\beta})^{1/\beta}.$$

It can be checked that this operation is binary and associative on the interval $A = (0, \infty)$ and the function associated with the binary associative operation * is $g(u) = u^{\beta}$. It is easy to check that

$$\frac{\bar{F}(t * u)}{\bar{F}(t)} = e^{-u^{\beta}}$$

which is strictly increasing in $0 \leq u < \infty$ for $\beta > 1$. Hence the Weibull distribution F has the IFR* property.

(ii)**Pareto distribution:** Suppose the random variable T follows a Pareto distribution

$$\bar{F}(u) = u^{-\alpha}, 1 \leq u < \infty, \alpha > 0.$$

Let $*$ be the binary associative opertion defined by

$$x * y = xy, x > 0, y > 0.$$

It can be checked that this operation is binary and associative on the interval $A = (0, \infty)$ and the function associated with the binary associative operation is $g(x) = -\log x$. Observe that

$$\frac{\bar{F}(t * u)}{\bar{F}(u)} = t^{-\alpha}$$

which is decreasing for $t \geq 1$. Hence the Pareto distribution has the DFR* property. **References**

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