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On equivariant Dirac operators for $SU_q(2)$

Partha Sarathi Chakraborty
Arupkumar Pal

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Dedicated to Prof. Kalyan Sinha on his sixtieth birthday.

Abstract

We give a decomposition of the spectral triple constructed by Dabrowski et al ([6]) in terms of the canonical equivariant spectral triple constructed by Chakraborty and Pal ([1]).

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1 Introduction

The interaction between noncommutative geometry and quantum groups, in particular the (noncommutative) geometry of quantum groups, had been one of the less understood and less explored areas of both the theories for a while. In the last few years, however, there have been some progress in this direction. The first important step was taken by the authors in [1] where they found an optimal family of Dirac operators for the quantum $SU(2)$ group acting on $L_2(h)$, the L_2 space of the Haar state h, and equivariant with respect to the (co-)action of the group itself. This family has quite a few remarkable features, e. g.

1. any element of the K-homology group can be realized by a member from this family, which means that all elements of the K-homology group are realizable through some Dirac operator acting on the single Hilbert space $L_2(h)$ in a natural manner,

2. the sign of any equivariant Dirac operator on $L_2(h)$ is a compact perturbation of the sign of a Dirac operator from this family,

3. given any equivariant Dirac operator $\widetilde{D}$ acting on $L_2(h)$, one can get a Dirac operator D from this family and two constants k_1 and k_2 such that

$$|\widetilde{D}| \leq k_1 + k_2|D|,$$

4. they exhibit features that are unique to the quantum case ($q \neq 1$). It was proved in [1] that for clasical $SU(2)$, there does not exist any Dirac operator acting on the L_2 space that is both equivariant as well as 3-summable.

These triples were later analysed by Connes ([4]) in great detail, where the general theory of Connes-Moscovici was applied to obtain a beautiful local index formula for $SU_q(2)$.

Recently, Dabrowski et al ([6]) have constructed another family of Dirac operators that act on two copies of the L_2 space, has the right summability property, is equivariant in a sense described in [6], and is isospectral to the Classical Dirac operator. In this note, we will give a decomposition of this Dirac operator in terms of the Dirac operators constructed in [1]. This suggests, among other things, that the triples constructed in [1] are also minimal in some sense.

2 Canonical triples for $SU_q(2)$

Let q be a real number in the interval $(0, 1)$. Let $\mathcal{A}$ denote the C^*-algebra of continuous functions on $SU_q(2)$, which is the universal C^*-algebra generated by two elements α and β subject to the relations

$$\alpha^*\alpha + \beta^*\beta = I = \alpha\alpha^* + q^2\beta\beta^*, \quad \alpha\beta - q\beta\alpha = 0 = \alpha\beta^* - q\beta^*\alpha, \quad \beta^*\beta = \beta\beta^*$$

as in [1]. Let Δ denote the comultiplication map, h the Haar state, Ω the cyclic vector in $L_2(h)$ and $\pi : \mathcal{A} \to \mathcal{L}(L_2(h))$ the representation given by left multiplication by elements in $\mathcal{A}$. Let u denote the right regular representation of $SU_q(2)$. Recall ([7]) that u is the unique representation acting on $L_2(h)$ that obeys the condition

$$\Big((\mathrm{id} \otimes \rho)u\Big)\pi(a)\Omega = \pi\Big((\mathrm{id} \otimes \rho)\Delta(a)\Big)\Omega \tag{2.1}$$

for all $a \in \mathcal{A}$ and for all continuous linear functionals ρ on $\mathcal{A}$. In [1], the authors studied right equivariant Dirac operators, those Dirac operators that commute with the right regular representation, i. e. D acting on $L_2(h)$ for which

$$(D \otimes I)u = u(D \otimes I).$$

In particular, an optimal family of equivariant Dirac operators were found. A generic member of this family is of the form

$$e_{ij}^{(n)} \mapsto \begin{cases} (an + b)e_{ij}^{(n)} & \text{if } -n \leq i < n - k, \\ (cn + d)e_{ij}^{(n)} & \text{if } i = n - k, n - k + 1, \ldots, n, \end{cases}$$

where k is a fixed nonnegative integer and a, b, c, d are reals with $ac < 0$. If one looks at left equivariant Dirac operators, the same arguments would then lead to the following proposition.

Proposition 2.1 *Let v be the left regular representation of $SU_q(2)$. Let k be a nonnegative integer and let a, b, c, d be real numbers with $ac < 0$. Then the operator $D \equiv D(k,a,b,c,d)$ on $L_2(h)$ given by*

$$e_{ij}^{(n)} \mapsto \begin{cases} (an+b)e_{ij}^{(n)} & \text{if } -n \leq j < n-k, \\ (cn+d)e_{ij}^{(n)} & \text{if } j = n-k, n-k+1, \ldots, n, \end{cases}$$

gives a spectral triple $(\pi, L_2(h), D)$ having nontrivial Chern character and obeys

$$(D \otimes I)v = v(D \otimes I). \tag{2.2}$$

Conversely, given any spectral triple $(\pi, L_2(h), \widetilde{D})$ with nontrivial Chern character such that $(\widetilde{D} \otimes I)v = v(\widetilde{D} \otimes I)$, there exist a nonnegative integer k and reals a, b, c, d with $ac < 0$ such that

1. *sign $\widetilde{D}$ is a compact perturbation of the sign of $D \equiv D(k,a,b,c,d)$, and*

2. *there exist constants k_1 and k_2 such that*

$$|\widetilde{D}| \leq k_1 + k_2 |D|.$$

Proof: The key point is to note that the characterizing property of the left regular representation v is

$$\Big((\mathrm{id} \otimes \rho)v^*\Big)\pi(a)\Omega = \pi\Big((\rho \otimes \mathrm{id})\Delta(a)\Big)\Omega. \tag{2.3}$$

Thus on the right hand side, one now has left convolution of a by ρ instead of right convolution by ρ. Therefore any self-adjoint operator on $L_2(h)$ with discrete spectrum that obeys $(D \otimes I)v = v(D \otimes I)$ will be of the form

$$e_{ij}^{(n)} \mapsto \lambda(n,j)e_{ij}^{(n)}.$$

Hence if one now proceeds exactly along the same lines as in [1], one gets all the desired conclusions. □

Observe at this point that the whole analysis carried out in [4] will go through for this Dirac operator as well. Let us now take two such Dirac operators D_1 and D_2 on $L_2(h)$ given by

$$D_1 e_{ij}^n = \begin{cases} -2ne_{ij}^n & \text{if } j \neq n, \\ (2n+1)e_{ij}^n & \text{if } j = n, \end{cases} \qquad D_2 e_{ij}^n = \begin{cases} (-2n-1)e_{ij}^n & \text{if } j \neq n, \\ (2n+1)e_{ij}^n & \text{if } j = n. \end{cases} \tag{2.4}$$

Now look at the triple $(L_2(h) \oplus L_2(h), \pi \oplus \pi, D_1 \oplus |D_2|)$. It is easy to see that this is a spectral triple. Nontriviality of its Chern character is a direct consequence of that of D_1.

We will show in the next section that the spectral triple constructed in [6] is nothing but this above triple.

3 The decomposition

Let us briefly recall the Dirac operator constructed in [1]. The carrier Hilbert space $\mathscr{H}$ is a direct sum of two copies of $L_2(h)$ that decomposes as

$$\mathscr{H} = W_0^{\uparrow} \oplus \Big(\bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} (W_n^{\uparrow} \oplus W_n^{\downarrow}) \Big),$$

where

$$\begin{aligned} W_n^{\uparrow} &= \operatorname{span}\{u_{ij}^n : i = -n, -n+1, \ldots, n,\ j = -n - \frac{1}{2}, -n + \frac{1}{2}, \ldots, n + \frac{1}{2}\}, \\ W_n^{\downarrow} &= \operatorname{span}\{d_{ij}^n : i = -n, -n+1, \ldots, n,\ j = -n + \frac{1}{2}, -n + \frac{3}{2}, \ldots, n - \frac{1}{2}\}. \end{aligned}$$

(u_{ij}^n and d_{ij}^n correspond to the basis elements $|nij\uparrow\rangle$ and $|nij\downarrow\rangle$ respectively in the notation of [6]). Now write

$$v_{ij}^n = \begin{pmatrix} u_{ij}^n \\ d_{ij}^n \end{pmatrix}$$

with the convention that $d_{ij}^n = 0$ for $j = \pm(n + \frac{1}{2})$. Then the representation π' of $\mathcal{A}$ on $\mathscr{H}$ is given by

$$\begin{aligned} \pi'(\alpha^*)\, v_{ij}^n &= a_{nij}^+ v_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} + a_{nij}^- v_{i+\frac{1}{2}, j+\frac{1}{2}}^{n-\frac{1}{2}}, \\ \pi'(-\beta)\, v_{ij}^n &= b_{nij}^+ v_{i+\frac{1}{2}, j-\frac{1}{2}}^{n+\frac{1}{2}} + b_{nij}^- v_{i+\frac{1}{2}, j-\frac{1}{2}}^{n-\frac{1}{2}}, \\ \pi'(\alpha)\, v_{ij}^n &= \tilde{a}_{nij}^+ v_{i-\frac{1}{2}, j-\frac{1}{2}}^{n+\frac{1}{2}} + \tilde{a}_{nij}^- v_{i-\frac{1}{2}, j-\frac{1}{2}}^{n-\frac{1}{2}}, \\ \pi'(-\beta^*)\, v_{ij}^n &= \tilde{b}_{nij}^+ v_{i-\frac{1}{2}, j+\frac{1}{2}}^{n+\frac{1}{2}} + \tilde{b}_{nij}^- v_{i-\frac{1}{2}, j+\frac{1}{2}}^{n-\frac{1}{2}}, \end{aligned}$$

where $a_{nij}^{\pm}$ and $b_{nij}^{\pm}$ are the following 2×2 matrices:

$$\begin{aligned} a_{nij}^+ &= q^{(i+j-\frac{1}{2})/2} [n+i+1]^{\frac{1}{2}} \begin{pmatrix} q^{-n-\frac{1}{2}} \frac{[n+j+\frac{3}{2}]^{1/2}}{[2n+2]} & 0 \\ q^{\frac{1}{2}} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]\,[2n+2]} & q^{-n} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]} \end{pmatrix}, \\ a_{nij}^- &= q^{(i+j-\frac{1}{2})/2} [n-i]^{\frac{1}{2}} \begin{pmatrix} q^{n+1} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]} & -q^{\frac{1}{2}} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n]\,[2n+1]} \\ 0 & q^{n+\frac{1}{2}} \frac{[n-j-\frac{1}{2}]^{1/2}}{[2n]} \end{pmatrix}, \\ b_{nij}^+ &= q^{(i+j-\frac{1}{2})/2} [n+i+1]^{\frac{1}{2}} \begin{pmatrix} \frac{[n-j+\frac{3}{2}]^{1/2}}{[2n+2]} & 0 \\ -q^{-n-1} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]\,[2n+2]} & q^{-\frac{1}{2}} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]} \end{pmatrix}, \\ b_{nij}^- &= q^{(i+j-\frac{1}{2})/2} [n-i]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]} & -q^{n} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n]\,[2n+1]} \\ 0 & -\frac{[n+j-\frac{1}{2}]^{1/2}}{[2n]} \end{pmatrix}, \end{aligned}$$

($[m]$ being the q-number $\frac{q^m - q^{-m}}{q - q^{-1}}$) and $\tilde{a}_{nij}^{\pm}$ and $\tilde{b}_{nij}^{\pm}$ are the hermitian conjugates of the above ones:

$$\tilde{a}_{nij}^{\pm} = (a_{n\pm\frac{1}{2}, i-\frac{1}{2}, j-\frac{1}{2}}^{\mp})^*, \qquad \tilde{b}_{nij}^{\pm} = (b_{n\pm\frac{1}{2}, i-\frac{1}{2}, j+\frac{1}{2}}^{\mp})^*.$$

The operator D is given by

$$Du^n_{ij} = (2n+1)u^n_{ij}, \qquad Dd^n_{ij} = -2nd^n_{ij}.$$

The triple $(\pi', \mathscr{H}, D)$ is precisely the triple constructed in [6].

Theorem 3.1 *Let $\mathcal{K}_q$ be the two-sided ideal of $\mathcal{L}(\mathscr{H})$ generated by the operator*

$$d^n_{ij} \mapsto q^n d^n_{ij}, \quad u^n_{ij} \mapsto q^n u^n_{ij},$$

*and let $\mathcal{A}_f$ denote the *-subalgebra of $\mathcal{A}$ generated by α and β. Then there is a unitary $U : L_2(h) \oplus L_2(h) \to \mathscr{H}$ such that*

$$\begin{aligned} U(D_1 \oplus |D_2|)U^* &= D, && (3.5)\\ U\Big(\pi(a) \oplus \pi(a)\Big)U^* - \pi'(a) &\in \mathcal{K}_q \qquad \textit{for all } a \in \mathcal{A}_f. && (3.6) \end{aligned}$$

Proof: Define $U : L_2(h) \oplus L_2(h) \to \mathscr{H}$ as follows

$$\begin{aligned} U(e^{(n)}_{ij} \oplus 0) &= d^n_{i,j+\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n, \; j = -n, -n+1, \ldots, n-1,\\ U(e^{(n)}_{in} \oplus 0) &= u^n_{i,n+\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n,\\ U(0 \oplus e^{(n)}_{ij}) &= u^n_{i,j-\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n, \; j = -n, -n+1, \ldots, n. \end{aligned}$$

It is immediate that $U(D_1 \oplus |D_2|)U^* = D$. Therefore all that we need to prove now is that $U(\pi(a) \oplus \pi(a))U^* - \pi'(a) \in \mathcal{K}_q$ for all $a \in \mathcal{A}_f$. For this, let us introduce the representation $\widehat{\pi} : \mathcal{A} \to \mathcal{L}(L_2(h))$ given by

$$\widehat{\pi}(\alpha) = \widehat{\alpha}, \qquad \widehat{\pi}(\beta) = \widehat{\beta},$$

where $\widehat{\alpha}$ and $\widehat{\beta}$ are the following operators on $L_2(h)$ (see lemma 2.2, [2])

$$\begin{aligned} \widehat{\alpha} : e^{(n)}_{ij} &\mapsto q^{2n+i+j+1} e^{(n+\frac{1}{2})}_{i-\frac{1}{2},j-\frac{1}{2}} + (1-q^{2n+2i})^{\frac{1}{2}}(1-q^{2n+2j})^{\frac{1}{2}} e^{(n-\frac{1}{2})}_{i-\frac{1}{2},j-\frac{1}{2}}, && (3.7)\\ \widehat{\beta} : e^{(n)}_{ij} &\mapsto -q^{n+j}(1-q^{2n+2i+2})^{\frac{1}{2}} e^{(n+\frac{1}{2})}_{i+\frac{1}{2},j-\frac{1}{2}} + q^{n+i}(1-q^{2n+2j})^{\frac{1}{2}} e^{(n-\frac{1}{2})}_{i+\frac{1}{2},j-\frac{1}{2}}, && (3.8) \end{aligned}$$

It is easy to see that

$$\pi(a) \oplus \pi(a) - \widehat{\pi}(a) \oplus \widehat{\pi}(a) \in U^* \mathcal{K}_q U.$$

for $a = \alpha^*$ and $a = \beta$. Therefore it is enough to verify that

$$U(\widehat{\pi}(a) \oplus \widehat{\pi}(a))U^* - \pi'(a) \in \mathcal{K}_q$$

for $a = \alpha^*$ and for $a = \beta$.

Next observe that

$$
\begin{aligned}
a^+_{nij} &= (1-q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} (1-q^{2n+2j+3})^{\frac{1}{2}} & 0 \\ 0 & (1-q^{2n+2j+1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}), \\
a^-_{nij} &= q^{2n+i+j+\frac{1}{2}}(1-q^{2n-2i})^{\frac{1}{2}} \begin{pmatrix} q(1-q^{2n-2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1-q^{2n-2j-1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}), \\
b^+_{nij} &= q^{n+j-\frac{1}{2}}(1-q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} q(1-q^{2n-2j+3})^{\frac{1}{2}} & 0 \\ 0 & (1-q^{2n-2j+1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}), \\
&= q^{n+j-\frac{1}{2}}(1-q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} + O(q^{2n}), \\
b^-_{nij} &= -q^{n+i}(1-q^{2n-2i})^{\frac{1}{2}} \begin{pmatrix} (1-q^{2n+2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1-q^{2n+2j-1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}) \\
&= -q^{n+i} \begin{pmatrix} (1-q^{2n+2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1-q^{2n+2j-1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}).
\end{aligned}
$$

The required result now follows from this easily. □

Remark 3.2 The above result implies in particular that any property of the Dirac operator constructed in [6] would essentially be a consequence of the corresponding property of the Dirac operators in [1]. For example, the nontriviality of D is now an easy consequence of that of $D_1 \oplus |D_2|$, which in turn comes from the nontriviality of D_1.

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Partha Sarathi Chakraborty (`chakrabortyps@cf.ac.uk`)

School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, UK

Arupkumar Pal (`arup@isid.ac.in`)

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA