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Summary. A popular measure to assess supersaturated designs is the $E(s^2)$ crietrion. In this paper, improved lower bounds on $E(s^2)$ are obtained. Examples of $E(s^2)$-optimal designs attaining the improved bounds are presented. The equivalence of the bounds obtained by Butler *et al.* (2001) (in the cases where their result applies) and those obtained by Bulutoglu and Cheng (2004) is established. The concept of near $E(s^2)$-optimal design is introduced and some means for arriving at such designs are suggested. We also give two simple methods of constructing $E(s^2)$-optimal designs.

Keywords: Effect sparsity; Lower bound; Screening designs.

1. Introduction

Supersaturated designs and their application in factor screening experiments have received considerable attention in the recent past. In a factorial experiment involving m two-level factors and n runs, n is required to be at least $m + 1$ for the estimability of all main effects. A design is called supersaturated if $n < m + 1$. Under the assumption of effect sparsity that only a small number of factors are active, a supersaturated design can provide considerable cost saving in factor screening.

We represent an n-run supersaturated design for m two-level factors by an $n \times m$ matrix X of 1's and -1's where we assume that each column of X has exactly $n/2$ 1's and and $n/2$ -1's and $n > 4$ is even. Also, we assume that for any two columns $\boldsymbol{u} = (u_1, \ldots, u_n)'$ and $\boldsymbol{v} = (v_1, \ldots, v_n)'$ of X, $\boldsymbol{u} \neq \pm\boldsymbol{v}$. The number of possible factors that can be accommodated is at most M, where $M = \frac{1}{2}\binom{n}{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1}$. Thus we have $n - 1 < m \leq M$. Supersaturated designs have mainly been constructed by using the $E(s^2)$-optimality criterion originally proposed by Booth and Cox (1962). An $E(s^2)$-optimal supersaturated design is one that minimizes $E(s^2) = \sum_{i \neq j} s_{ij}^2/\{m(m-1)\}$, where s_{ij} is the (i,j)-th entry of $X'X$.

Nguyen (1996) and Tang and Wu (1997) independently derived the lower bound

$$E(s^2) \geq \frac{(m-n+1)n^2}{(m-1)(n-1)}, \tag{1.1}$$

for any supersaturated design with m factors and n runs. When $n \equiv 0 (\text{mod } 4)$, this bound can be achieved only if m is a multiple of $n-1$; when $n \equiv 2 (\text{mod } 4)$, m needs to be an even multiple of $n-1$. Butler *et al.* (2001) and Bulutoglu and Cheng (2004) provided better lower bounds for $E(s^2)$ than (1.1).

In Section 2, we obtain further improved lower bounds on $E(s^2)$. Examples of $E(s^2)$-optimal designs attaining this improved bound are presented. In Section 3, we present in simpler terms the improved lower bounds. In the process, the equivalence of the bounds obtained by Butler *et al.* (2001) (in the cases where their result applies) and those obtained by Bulutoglu and Cheng (2004) is established. $E(s^2)$-optimal designs attaining the lower bounds are not always available and in such situations, one can consider designs that are nearly $E(s^2)$-optimal. We give a formal definition of near $E(s^2)$-optimal design and suggest some means for arriving at such designs. Finally in Section 4, we give two simple methods of constructing $E(s^2)$-optimal designs. Eskridge *et al.* (2004) obtained near $E(s^2)$-optimal designs for $n \equiv 2 (\text{mod } 4)$ and $m = j(n-1)$, j odd, using the properties of regular graph designs. We show that these designs have $E(s^2)$-efficiency greater than 0.9774, which is much higher than what is presently known.

2. Improved lower bounds on $E(s^2)$

From Theorem 3.1 of Bulutoglu and Cheng (2004), it follows that for given m and n, $m > n-1$, (m not a multiple of $n-1$ when $n \equiv 0 (\text{mod } 4)$; or, m not an even multiple of $n-1$ when $n \equiv 2 (\text{mod } 4)$) there exists a unique integer q such that $(q-2)(n-1) < m < (q+2)(n-1)$ and $(m+q) \equiv 2 (\text{mod } 4)$. However, if we do not put the restriction of (i) m not being a multiple of $n-1$ when $n \equiv 0 (\text{mod } 4)$ and (ii) m not being an even multiple of $n-1$ when $n \equiv 2 (\text{mod } 4)$, then we show that there exists a unique non-negative integer q such that $(q-2)(n-1) \leq m < (q+2)(n-1)$ and $(m+q) \equiv 2 (\text{mod } 4)$. An explicit expression for q is also given. In can be verified that Bulutoglu-Cheng's proof goes through even for $m = n-1$ and for q such that $(q-2)(n-1) \leq m < (q+2)(n-1)$ and $(m+q) \equiv 2 (\text{mod } 4)$. Thus, subject to this minor modification, Theorem 3.1 of Bulutoglu and Cheng (2004) would hold for all m and n, with $m \geq n-1$. Note that though supersaturated designs have been defined only for $m > n-1$, the bounds are still meaningful for $m \geq n-1$. For given m and n, the following result gives an explicit expression for q, a proof of which is given in the Appendix. Throughout, for $z > 0$, $[z]$ stands for the largest integer contained in z.

Lemma 2.1. For given $n \equiv 0 (\text{mod } 2)$, $m \equiv k (\text{mod } 4)$ and $m \geq n-1$, there is a unique non-negative integer q such that $(q-2)(n-1) \leq m < (q+2)(n-1)$ and $(m+q) \equiv 2 (\text{mod } 4)$. This unique q is given by $q = 4[\frac{m+k(n-1)}{4(n-1)}] + 2 - k$.

Let

$$
\begin{array}{lll}
g(q) = n((m+q)^2 - n(q^2+m)), & a_1 = (q-2)(n-1), & a_2 = (q-2)(n-1) + n/2, \\
a_3 = (q-1)(n-1), & a_4 = (q+1)(n-1), & a_5 = (q+2)(n-1) - n/2, \\
a_6 = (q+2)(n-1), & a_2' = (q-3)(n-1) + 3n/2, & a_5' = (q+3)(n-1) - 3n/2, \\
a_2'' = (q-1)(n-1) - n/2, & a_5'' = (q+1)(n-1) + n/2. &
\end{array}
$$

Also, let

$$
\begin{aligned}
\mathcal{R} &= \{m : \ a_1 \le m < a_6\}, \\
\mathcal{R}_{11} &= \{m : \ a_3 \le m \le a_4\} = \mathcal{R}_{21} = \mathcal{R}_{31}, \\
\mathcal{R}_{12} &= \{m : \ a_2 \le m < a_3 \ \text{ or, } \ a_4 < m \le a_5\}, \\
\mathcal{R}_{13} &= \{m : \ a_1 \le m < a_2 \ \text{ or, } \ a_5 < m < a_6\}, \\
\mathcal{R}_{22} &= \{m : \ a_2' \le m < a_3 \ \text{ or, } \ a_4 < m \le a_5'\}, \\
\mathcal{R}_{23} &= \{m : \ a_1 \le m < a_2' \ \text{ or, } \ a_5' < m < a_6\}, \\
\mathcal{R}_{32} &= \{m : \ a_2'' \le m < a_3 \ \text{ or, } \ a_4 < m \le a_5''\}, \\
\mathcal{R}_{33} &= \{m : \ a_1 \le m < a_2'' \ \text{ or, } \ a_5'' < m < a_6\}.
\end{aligned}
$$

Note that $\mathcal{R}$ covers the entire range of m and

$$\mathcal{R} = \mathcal{R}_{11} \cup \mathcal{R}_{12} \cup \mathcal{R}_{13} = \mathcal{R}_{21} \cup \mathcal{R}_{22} \cup \mathcal{R}_{23} = \mathcal{R}_{31} \cup \mathcal{R}_{32} \cup \mathcal{R}_{33}.$$

Bulutoglu and Cheng (2004) in their Theorem 3.1 had considered the set $\{m : a_3 < m < a_4\}$ instead of the sets $\mathcal{R}_{i1}, \ i = 1, 2, 3$. However, since $m+q$ is even, $m \neq (q \pm 1)(n-1)$. Therefore, in the definition of $\mathcal{R}_{i1}, \ i = 1, 2, 3$, we have kept the closed intervals as above for m, without affecting the results.

The bounds as given in Theorem 3.1 of Bulutoglu and Cheng (2004) can now be rephrased as below.

For $i = 1, 2, 3$, when $n \equiv 0 (\text{mod } 4)$ and $m \in \mathcal{R}_{1i}$ the bound is

$$E(s^2) \ge \frac{B_{1i}}{m(m-1)},$$

where
$B_{11} = g(q) + 2n(n-2)$, $B_{12} = g(q) - 2n(n-2) + 4n|m - q(n-1)|$, $B_{13} = g(q) + 4n(n-1)$.

For $i = 1, 2, 3$, when $n \equiv 2 (\text{mod } 4)$, $m \in \mathcal{R}_{2i}$ and q is even, the bound is

$$E(s^2) \ge \max\left(\frac{B_{2i}}{m(m-1)}, 4\right),$$

where $B_{21} = g(q) + 2n(n-2) + 8$, $B_{22} = g(q) - 2n(n-10) + 4(n-2)|m - q(n-1)| - 24$, $B_{23} = g(q) + 4n(n-1)$.

For $i = 1, 2, 3$, when $n \equiv 2 (\text{mod } 4)$, $m \in \mathcal{R}_{3i}$ and q is odd, the bound is

$$E(s^2) \ge \max\left(\frac{B_{3i}}{m(m-1)}, 4\right),$$

where $B_{31} = g(q) + 2n(n-2)$, $B_{32} = g(q) - 2n(n-2) + 4n|m - q(n-1)|$, $B_{33} = g(q) + 4n(n-3) + 8|m - q(n-1)| + 8$.

Butler *et al.* (2001) as well as Bulutoglu and Cheng (2004), while obtaining the lower bounds for $E(s^2)$ used structural properties of the matrix XX'. We use a property of the matrix $X'X$ to improve the above mentioned lower bounds of $E(s^2)$. We shall use $A(v)$ to denote a term which has a factor v. The following three Lemmas are useful in the sequel. The proofs of these lemmas appear in the Appendix.

Lemma 2.2. For $n \equiv 0 \pmod 4$, each of B_{11}, B_{12} and B_{13} is a multiple of 32.

Lemma 2.3. For $n \equiv 2 \pmod 4$ and q even, each of $B_{21} - 4m(m-1)$, $B_{22} - 4m(m-1)$ and $B_{23} - 4m(m-1)$ is a multiple of 64.

Lemma 2.4. For $n \equiv 2 \pmod 4$ and q odd, $B_{32} - 4m(m-1)$ and $B_{33} - 4m(m-1)$ have a factor 64 whereas $B_{31} - 4m(m-1)$ has a factor 32. Moreover, $B_{31} - 4m(m-1)$ has a factor 64 unless (i) $m \equiv 1 \pmod 4$ and $(m+q) \equiv 6 \pmod 8$ or (ii) $m \equiv 3 \pmod 4$ and $(m+q) \equiv 2 \pmod 8$.

Based on the above Lemmas, one can prove the following result. For a proof, see the Appendix.

Theorem 2.1. Let $n \equiv 0 \pmod 2$, $m \equiv k \pmod 4$, $m \geq n-1$, $q = 4[\frac{m+k(n-1)}{4(n-1)}] + 2 - k$ and $g(q) = n((m+q)^2 - n(q^2+m))$. Then,

(1) when $n \equiv 0 \pmod 4$ and $m \in \mathcal{R}_{1i}$, a lower bound is $E(s^2) \geq B_{1i}/(m^2-m)$, $i = 1,2,3$, where $B_{11} = g(q) + 2n(n-2)$, $B_{12} = g(q) - 2n(n-2) + 4n|m - q(n-1)|$, $B_{13} = g(q) + 4n(n-1)$;

(2) when $n \equiv 2 \pmod 4$, $m \in \mathcal{R}_{2i}$ and q is even, a lower bound is $E(s^2) \geq \max(B_{2i}/(m^2-m), 4)$, $i = 1,2,3$, where $B_{21} = g(q) + 2n(n-2) + 8$, $B_{22} = g(q) - 2n(n-10) + 4(n-2)|m - q(n-1)| - 24$, $B_{23} = g(q) + 4n(n-1)$;

(3) when $n \equiv 2 \pmod 4$, $m \in \mathcal{R}_{3i}$ and q is odd, a lower bound is $E(s^2) \geq \max(B_{3i}/(m^2-m), 4)$, $i = 1,2,3$, where $B_{31} = g(q) + 2n(n-2) + x$, $B_{32} = g(q) - 2n(n-2) + 4n|m - q(n-1)|$, $B_{33} = g(q) + 4n(n-3) + 8|m - q(n-1)| + 8$ and

$$x = \begin{cases} 32 & \text{if } m \equiv (1+2j) \pmod 4 \text{ and } (m+q) \equiv (6-4j) \pmod 8, \text{ for } j = 0 \text{ or } 1, \\ 0 & \text{if } m \equiv (1+2j) \pmod 4 \text{ and } (m+q) \equiv (2+4j) \pmod 8, \text{ for } j = 0 \text{ or } 1. \end{cases}$$

Table 2.1 shows values of the Bulutoglu-Cheng bound and our improved bound for $n \leq m \leq 3(n-1)$, where $10 \leq n \leq 18$. In Table 2.2 we present some *new* 10- and 14-run $E(s^2)$-optimal designs, which includes designs attaining our improved bounds. For $n = 10$ and $m \leq 24$ (with the exception of $m = 12, 17, 18, 19, 24$), $E(s^2)$-optimal designs attaining the lower bounds of Theorem 2.1 are presented, where the columns to the right of m are excluded. The designs with $m = 12, 17, 19, 24$ are $NE(s^2)$-optimal (for a definition of $NE(s^2)$-optimality, see Section 3). However, $E(s^2)$-optimal designs for $m = 17, 18, 19$ can be constructed using the results of Nguyen (1996) and Cheng(1997). Similarly, the $E(s^2)$-optimal design with $n = 14$ and $m = 21$

is presented. Instead of providing the full X-matrix with elements 1 and -1, to save space, we indicate the $n/2$ runs under each factor which has a positive level $(+1)$.

Table 2.1. Comparison of bounds

n	m	Bound of Theorem 2.1	Bulutoglu-Cheng Bound
10	13	4.82051	4.61538
10	23	7.41502	7.35177
14	21	6.74285	6.66666
14	31	9.36774	9.33333
14	39	10.5641	10.54251
18	21	5.21904	5.14285
18	29	8.72906	8.68965
18	39	11.34143	11.31984
18	47	12.70305	12.68825

Table 2.2. $E(s^2)$-optimal designs

$n = 10, m \leq 24$ (except for $m = 12, 17, 18, 19, 24$)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	3	4	5	6	7	9	2	3	1	1	1	1	4	2	2	3	3	2	2	4	5	1	1
2	4	5	6	7	8	1	3	5	2	7	3	2	5	3	3	4	5	4	6	6	7	2	2
5	7	8	9	1	2	4	8	6	3	8	5	6	6	4	6	6	7	5	7	8	8	3	4
7	9	1	2	3	4	6	9	8	4	9	9	8	7	5	7	8	8	9	9	9	9	9	8
10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10

Table 2.2. (contd.)

$n = 14, m = 21$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	2	1	1	1	4	1	2	2	3	1	1	1	3	2	4	1	1	3	2	2
2	7	2	4	3	7	2	4	4	5	5	5	3	4	3	6	2	6	5	5	3
3	8	3	5	6	8	6	6	5	6	7	9	4	5	5	8	4	7	8	6	9
8	9	4	6	7	11	10	9	7	8	8	10	7	10	6	9	5	10	9	7	11
10	10	8	8	9	12	11	12	10	10	12	12	9	11	7	10	9	11	11	8	12
12	11	13	11	13	13	13	13	13	13	13	13	10	12	12	12	11	12	13	9	13
14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14

3. Equivalent form of the improved bounds

Butler *et al.* (2001) were first to obtain lower bounds for the cases when m is not a multiple of $n-1$. To be specific, Butler *et al.* obtained lower bounds to $E(s^2)$ for $m = p(n-1) \pm r$, $0 \le r < n/2$, where (i) p is positive and $n \equiv 0 \pmod 4$ and (ii) p is even and $n \equiv 2 \pmod 4$.

Bulutoglu and Cheng (2004) made a numerical comparison to see how their bounds compare with those of of Butler *et al.* (2001). The numerical comparison suggested that Bulutoglu-Cheng bounds are in agreement with Butler *et al.* bounds in the cases where they are applicable. We now give an equivalent form of our improved lower bounds. This equivalent form also establishes the equivalence of the bounds obtained by Bulutoglu and Cheng (2004) and those obtained earlier by Butler *et al.* (2001) for all cases where their result applies. In the process, we present in simpler terms the lower bounds of Bulutoglu and Cheng (2004) and their improvements. We have the following result for the improved lower bounds covering the full scenario in a more elegant form which in particular, includes the case of p being odd, a case not covered explicitly by the earlier results.

Theorem 3.1. For a supersaturated design with n runs and $m = p(n-1) \pm r$ factors (p positive, $0 \le r < \frac{n}{2}$), $E(s^2)$ is greater than or equal to the lower bound LB, where LB is as defined below:

(1) Let $n \equiv 0 (\text{mod } 4)$. Then,

$$LB = \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)} \left\{ D(n,r) - \frac{r^2}{n-1} \right\},$$

where

$$D(n,r) = \begin{cases} n+2r-3 & \text{for } r \equiv 1 (\text{mod } 4), \\ 2n-4 & \text{for } r \equiv 2 (\text{mod } 4), \\ n+2r+1 & \text{for } r \equiv 3 (\text{mod } 4), \\ 4r & \text{for } r \equiv 0 (\text{mod } 4). \end{cases}$$

(2) Let $n \equiv 2 (\text{mod } 4)$. Then,

$$LB = \max\left\{\frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)}\left\{D(n,r) - \frac{r^2}{n-1}\right\}, 4\right\},$$

where

(i) when p is even,

$$D(n,r) = \begin{cases} n + 2r - 3 + x/n & \text{for } r \equiv 1 (\text{mod } 4), \\ 2n - 4 + 8/n & \text{for } r \equiv 2 (\text{mod } 4), \\ n + 2r + 1 & \text{for } r \equiv 3 (\text{mod } 4), \\ 4r & \text{for } r \equiv 0 (\text{mod } 4), \end{cases}$$

(ii) when p is odd,

$$D(n,r) = \begin{cases} 2r - 8r/n + n - 16/n + 9 & \text{for } r \equiv 1 (\text{mod } 4), \\ 4r - 8r/n - 8/n + 8 & \text{for } r \equiv 2 (\text{mod } 4), \\ 2r + n + 8/n - 3 & \text{for } r \equiv 3 (\text{mod } 4), \\ 2n - 4 + x/n & \text{for } r \equiv 0 (\text{mod } 4), \end{cases}$$

and $x = 32$ if $\left\{\frac{m-1-2i}{4} + [\frac{m+(1+2i)(n-1)}{4(n-1)}]\right\} \equiv (1-i)(\text{mod } 2)$, for $i = 0$ or 1; else $x = 0$.

A proof of Theorem 3.1 appears in the Appendix. Note that LB in (1) of Theorem 3.1 is the same as in Butler *et al.* (2001) while those in (2) are improvements over the earlier bounds.

$E(s^2)$-optimal designs, attaining the lower bound, do not exist in all situations and in such a case, one might look for a design that is close to an $E(s^2)$-optimal design. We call a design nearly $E(s^2)$-optimal ($NE(s^2)$-optimal) if either of the following holds:
(i) $n \equiv 0(\text{mod } 4)$ and $E(s^2) = LB + 32/(m^2 - m)$,
(ii) $n \equiv 2(\text{mod } 4)$ and $E(s^2) = LB + 64/(m^2 - m)$,
where LB is as in Theorem 3.1.

Table 2.2 gives $NE(s^2)$-optimal designs for $n = 10$ with $m = 12, 17, 19, 24$. However, note that $E(s^2)$-optimal designs for $m = 17, 19$ can be constructed using the results of Nguyen (1996) and Cheng(1997). Also, for $n = 10$ with $m = 12$, Table 3.1 gives an $E(s^2)$-optimal design. Similarly, for $n = 14$ and $m = 18, 19, 20$, Table 2.2 gives $NE(s^2)$-optimal designs where, as indicated earlier, the columns to the right of m are excluded. In this case too, $E(s^2)$-optimal designs for $n = 14$ and $m = 18, 19$ have been identified in Bulutoglu and Cheng (2004). In general, computer software Gendex developed by Nguyen (1996) can be used to identify $E(s^2)$- and $NE(s^2)$-optimal designs.

Table 3.1. $E(s^2)$-optimal design

$n = 10, m = 12$

1	2	3	4	5	6	7	8	9	10	11	12
4	2	3	4	1	1	2	1	3	1	2	1
5	4	4	5	2	6	3	3	5	2	7	5
7	6	7	6	3	8	6	4	8	4	8	6
8	8	9	9	5	9	7	8	9	9	9	7
10	10	10	10	10	10	10	10	10	10	10	10

4. Methods of constructing $E(s^2)$-optimal designs

In this section, we give two methods for constructing $E(s^2)$-optimal supersaturated designs. In the first method, Hadamard matrices are used to obtain $E(s^2)$-optimal designs for 2^{n+1} or 2^n experiments in n runs where $n \equiv 2 \pmod 4$. In the second method we use complement of a supersaturated design and show that the complementary design is $E(s^2)$-optimal if the original supersaturated design is $E(s^2)$-optimal. This idea exists in, for example, Bulutoglu and Cheng (2004) and Eskridge *et al.* (2004), in their construction of $E(s^2)$-optimal designs attaining the lower bound (1.1). However, in their case, this property was attributed to augmentation of two balanced incomplete block designs. Our result generalizes the idea to cover all cases. In fact, this result reduces the general problem of identifying $E(s^2)$-optimal designs to half. That is, one needs only to look for $E(s^2)$-optimal designs with $m \le \frac{1}{4}\binom{n}{\frac{n}{2}} = \frac{M}{2}$. Eskridge *at al.* (2004) advocated an approach for constructing $E(s^2)$-optimal designs with a large number of factors. Since our result shows the equivalence of the $E(s^2)$-optimality property of a supersaturated design and its complement, the designs of Eskridge *et al.* (2004) may as well be used to generate new $E(s^2)$-optimal or near optimal designs. We present the two methods of construction below, the proofs of which can be found in the Appendix.

Theorem 4.1. Let $n \equiv 2 \pmod 4$. Then the existence of a Hadamard matrix of order $n + 2$ implies that of a supersaturated design X and,
(1) X is $E(s^2)$-optimal for a 2^{n+1} experiment in n runs,
(2) the design remaining after deleting any one column of X is $E(s^2)$-optimal for a 2^n experiment in n runs.

Note that since for a 2^{n+1} experiment in n runs ($n \equiv 2 \pmod 4$), the design X has $E(s^2) = 4$, therefore any subset of the columns of X would give rise to a design having minimum $E(s^2)$.

Theorem 4.2. Let d be an $E(s^2)$-optimal supersaturated design for a 2^m experiment in n runs, where $n - 1 < m \le M$. Then the design d' having $M - m$ columns of Y which are not columns (or -1 times the column) of d is an $E(s^2)$-optimal supersaturated design for a 2^{M-m} experiment in n runs, where Y is a $n \times M$ matrix, whose columns represent the M factors, such that for any two columns $\boldsymbol{u} = (u_1, \ldots, u_n)'$ and $\boldsymbol{v} = (v_1, \ldots, v_n)'$ of Y, $\boldsymbol{u} \ne \pm\boldsymbol{v}$.

Based on the above result, it would be natural to ask whether the relation between the $E(s^2)$ values of the original design g and its complement g', namely

$$E_{g'} = \frac{n^2(M-2m)(M-n+1)}{(n-1)(M-m)(M-m-1)} + \frac{m(m-1)E_g}{(M-m)(M-m-1)} \tag{4.1}$$

(*vide* (A.26) of the Appendix) can be used to further improve the bounds of Theorem 3.1. In what follows, we show that the lower bounds of Theorem 3.1 agree (except when $n \equiv 2(\text{mod } 4)$ with either $m \leq n+1$ or $m \geq M-n-1$) for the designs g' when one uses the lower bound of Theorem 3.1 for g in (4.1). In other words, the lower bound for g' can simply be obtained by substituting the lower bound for g in (4.1).

Firstly, note that when $n \equiv 2(\text{mod } 4)$,

$$\frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)}\left\{D(n,r) - \frac{r^2}{n-1}\right\} \begin{array}{lll} < & 4 & \text{when } m < n+2, \\ = & 4 & \text{when } m = n+2, \\ > & 4 & \text{when } m > n+2. \end{array}$$

We can write M as $M = p^*(n-1)$, where $p^* = t^{-1}\binom{2(t-1)}{t-1}$ is a positive integer (being a Catalan number). Now, for design g, let $m = p(n-1) \pm r$ (p positive, $0 \leq r < \frac{n}{2}$, $m \geq n+2$). Then, for g' the number of factors is $M-m = M-p(n-1) \mp r = (p^*-p)(n-1) \mp r$. Also, when t is odd (i.e., the case when $n \equiv 2(\text{mod } 4)$), by taking $t = 2w+1$ for some w, $p^* = (2w+1)^{-1}\binom{4w}{2w}$ is even since $\binom{4w}{2w}$ is even. Thus, when $n \equiv 2(\text{mod } 4)$, for p even (odd), $p^* - p$ is even (odd). Therefore, while obtaining LB using Theorem 3.1, the values of $\left\{D(n,r) - \frac{r^2}{n-1}\right\} = H$ (say), are same for g and g'. Let g attain the lower bound LB. Then, substituting the LB value in place of E_g in (4.1) gives (on simplification),

$$E_{g'} = \frac{n^2(M-m-n+1)}{(n-1)(M-m-1)} + \frac{nH}{(M-m)(M-m-1)}.$$

This establishes our contention.

When $n \equiv 2(\text{mod } 4)$, $LB = 4$ for $m \leq n+2$ and $LB > 4$ for $m > n+2$. This is the reason why, for $n \equiv 2(\text{mod } 4)$ with $m \leq n+1$, on substituting $E_g = 4$, (4.1) gives $E_{g'}$ (with $M-m$ factors) leading to a lower bound sharper than what Theorem 3.1 gives.

Based on Theorem 4.2 and the above discussion, new $E(s^2)$- and $NE(s^2)$-optimal designs for $n = 10$ and $n = 14$ can be constructed as complements of the designs given in Tables 2.2 and 3.1.

Eskridge *et al.* (2004) obtained near optimal designs for $n \equiv 2(\text{mod } 4)$ and $m = j(n-1)$, j odd, using the properties of regular graph designs (RGD). For their designs obtained from generators of cyclic RGD,

$$E(s^2) = \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{4(n-1)(n-2)}{m(m-1)}. \tag{4.2}$$

With $n \geq 10$ and $j > 2$, Eskridge *et al.* (2004) obtained the lower bounds to the $E(s^2)$-efficiency (using Nguyen-Tang-Wu bounds) and also showed that their designs have $E(s^2)$-efficiency greater than 0.9493. Now, for $n \equiv 2 (\mathrm{mod}\ 4)$, $m = j(n-1)$, j odd, based on Theorem 3.1, the lower bound for $E(s^2)$ is given as

$$E(s^2) \geq \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n(2n-4+x/n)}{m(m-1)}, \tag{4.3}$$

where $x = 32$ if $\left\{\frac{m-1-2i}{4} + [\frac{m+(1+2i)(n-1)}{4(n-1)}]\right\} \equiv (1-i)(\mathrm{mod}\ 2)$, for $i = 0$ or 1; else $x = 0$.

Using (4.2) and (4.3) we get a sharper lower bound to the $E(s^2)$-efficiency, given by

$$E(s^2)\text{-efficiency} \geq \frac{1+b}{1+a}, \tag{4.4}$$

where $a = 4(n-2)(n-1)^2/\{mn^2(m-n+1)\}$, $b = (2n-4+x/n)(n-1)/\{mn(m-n+1)\}$ and $x = 32$ if $\{(m-1-2i)/4 + [\{m+(1+2i)(n-1)\}/\{4(n-1)\}]\} \equiv (1-i)(\mathrm{mod}\ 2)$, for $i = 0$ or 1; else $x = 0$.

From (4.4), it follows that the designs based on an RGD have $E(s^2)$-efficiency greater than 0.9774. This shows that RGD based designs have efficiencies higher than what is presently known. Similarly, the lower bounds to $NE(s^2)$-efficiency can be worked out which indicates that designs based on an RGD have $NE(s^2)$-efficiency greater than 0.9868.

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Appendix

A.1. Proof of Lemma 2.1.

Given m, $q = 4x + 2 - k$ for some integer x, since $m \equiv k(\mathrm{mod}\ 4)$ and $(m+q) \equiv 2(\mathrm{mod}\ 4)$. Now, $(q-2)(n-1) \leq m < (q+2)(n-1)$ implies $m/(n-1) - 2 < q \leq m/(n-1) + 2$ which, on substituting for q, yields $\{m + k(n-1)\}/\{4(n-1)\} - 1 < x \leq \{m + k(n-1)\}/\{4(n-1)\}$. Thus, $x = [\{m+k(n-1)\}/\{4(n-1)\}]$ and we get the desired expression for q. By substituting the four possible values of k in the expression for q, it follows that q is non-negative.

A.2. Proof of Lemma 2.2.

For $t > 0$, let $n = 4t$. Now, since $(m+q) \equiv 2 \pmod 4$, it follows that $q^2 + m$ is even. Let $q + m = 4w + 2$ for some positive w. Then
$g(q) = n((m+q)^2 - n(q^2+m)) = 4t(4w+2)^2 - 16t^2 A(2) = 64tw(w+1) + 16t - A(32)$
$= A(32) + 16t$. Therefore,
$B_{11} = g(q) + 2n(n-2) = A(32) + 16t + 32t^2 - 16t = A(32)$, $B_{12} = g(q) - 2n(n-2) + 4n|m - q(n-1)| = A(32) + 16t - 32t^2 + 16t + 16t|2(2w - 2tq + 1)| = A(32)$, $B_{13} = g(q) + 4n(n-1) = A(32) + 16t + 64t^2 - 16t = A(32)$.

A.3. Proof of Lemma 2.3.

For $t > 0$, let $n = 4t+2$. Now, since $(m+q) \equiv 2 \pmod 4$ and q is even, it follows that either (i) $m \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, or (ii) $m \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$. This implies that for some nonnegative integers y, s, and $i = 0$ or 1, we have $m = 4y + 2i$ and $q = 4s + 2 - 2i$. Now, substituting for n, m and q, and using the fact $i^2 = i$, we have (after simplification)
$g(q) - 4m(m-1) = n((m+q)^2 - n(q^2+m)) - 4m(m-1) = 32(2t+1)\{y(y+1) - s(s+1) - 4ts(s+1) + 2ys - 4ts\} - 64\{t^2 + y^2 + yt(t+1)\} - 48t - 8 + 64i\{4st(t+1) + y + s\} + 32it(t+1) = A(64) - 48t - 8$.
Therefore,
$B_{21} - 4m(m-1) = g(q) - 4m(m-1) + 2n(n-2) + 8 = A(64) - 48t - 8 + 16t(2t+1) + 8 = A(64) + 32t(t-1) = A(64)$, $B_{22} - 4m(m-1) = g(q) - 4m(m-1) - 2n(n-10) + 4(n-2)|m - q(n-1)| - 24$
$= A(64) - 48t - 8 - 32t^2 + 48t + 8 \pm 64t(y - s - 4st + 2t(i-1) + i) \mp 32t = A(64) - 32t(t \pm 1) = A(64)$,
$B_{23} - 4m(m-1) = g(q) - 4m(m-1) + 4n(n-1) = A(64) - 48t - 8 + 8(8t^2 + 6t + 1) = A(64) + 64t^2 = A(64)$.

A.4. Proof of Lemma 2.4.

For $t > 0$, let $n = 4t + 2$. Now, since $(m+q) \equiv 2 \pmod 4$ and q odd, it follows that either (i) $m \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$, or (ii) $m \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$. This implies that for some nonnegative integers y, s, and $i = 0$ or 1, we have $m = 4y + 2i + 1$ and $q = 4s + 2i + 1$. Now, substituting for n, m and q, and using the fact $i^2 = i$, we have after simplification,
$g(q) - 4m(m-1) = n((m+q)^2 - n(q^2+m)) - 4m(m-1) = 64(2t+1)(ys + yi - ts - 2ts^2 - 2tsi) - 64t(y^2 - s^2 - ty - 2ti) - 64yi - 32it(t+1) - 32(y^2 + s^2 + t^2) - 16t = A(64) - 32(y^2 + s^2 + t^2) - 16t$.
Therefore,
$B_{31} - 4m(m-1) = g(q) - 4m(m-1) + 2n(n-2) = A(64) - 32(y^2 + s^2 + t^2) - 16t + 32t^2 + 16t = A(64) - 32(y^2 + s^2)$, $B_{32} - 4m(m-1) = g(q) - 4m(m-1) - 2n(n-2) + 4n|m - q(n-1)|$
$= A(64) - 32(y^2 + s^2 + t^2) - 16t - 32t^2 - 16t + 32(2t+1)|y - s - t - 2t(2s+i)| = A(64) - 64t(t - |y - s - t - 2t(2s+i)|) - 32\{y(y \mp 1) + s(s \pm 1) \pm 2t(2s+i) + t \pm t\} = A(64)$, $B_{33} - 4m(m-1) = g(q) - 4m(m-1) + 4n(n-3) + 8|m - q(n-1)| + 8 = A(64) - 32(y^2 + s^2 + t^2) - 16t + 64t^2 + 16t + 32|y - s - t - 2t(2s+i)|$
$= A(64) + 64t^2 - 32\{y(y \mp 1) + s(s \pm 1) + t(t \pm 1) \pm 2t(2s+i)\} = A(64)$.
Now, $B_{31} - 4m(m-1)$ is an odd multiple of 32 if and only if $y^2 + s^2$ is odd. Also, $y^2 + s^2$ is odd if and only if $y + s$ is odd. Let $y + s = 2w + 1$ for some w. Then $m + q = 4(y+s) + 2 + 4i = 8w + 6 + 4i$

and either of the following holds:
(i) $i = 0$, $(m+q) \equiv 6(\text{mod } 8)$,
(ii) $i = 1$, $(m+q) \equiv 2(\text{mod } 8)$.
Similarly, it follows that $y + s$ is even when $m + q = 8w + 2 + 4i$, implying either (i) $i = 0$, $(m+q) \equiv 2(\text{mod } 8)$, or (ii) $i = 1$, $(m+q) \equiv 6(\text{mod } 8)$.

A. 5. Proof of Theorem 2.1.
The result basically follows from Lemmas 2.1- 2.4 and the following facts:

(i) For $n \equiv 0(\text{mod } 4)$, s_{ij} is an integral multiple of 4 for all $i \neq j$. Therefore, $\sum_{i\neq j} s_{ij}^2$ is a multiple of 32.

(ii) For $n \equiv 2(\text{mod } 4)$, we have $|s_{ij}| \equiv 2(\text{mod } 4)$. This means that $s_{ij}^2 = A(32) + 4$. Therefore, $\sum_{i\neq j} s_{ij}^2 - 4m(m-1) = A(64)$.

For $n \equiv 0(\text{mod } 4)$, we have already seen in Lemma 2.2 that each of B_{11}, B_{12} and B_{13} is a multiple of 32. Also, when $n \equiv 2(\text{mod } 4)$, Lemmas 2.3 and 2.4 show that each of $B_{21} - 4m(m-1), B_{22} - 4m(m-1), B_{23} - 4m(m-1), B_{32} - 4m(m-1)$ and $B_{33} - 4m(m-1)$ is a multiple of 64 while $B_{31} - 4m(m-1)$ is a multiple of 32 but not necessarily a multiple of 64. This leads to the lower bound improvement for the bound involving B_{31} by increasing B_{31} to $B_{31} + 32$ in all those cases where $B_{31} - 4m(m-1)$ is not a multiple of 64.

A. 6. Proof of Theorem 3.1.
For some integers i and r, let $m = (q \mp i)(n-1) \pm r$. Then,

$$q = \frac{m \mp r}{n-1} \pm i \quad \text{and} \quad m + q = (q \mp i)n \pm (i + r) \equiv 2(\text{mod } 4). \tag{A.1}$$

Substituting the value of q from (A.1) in $g(q)$ and after some simplification, we have

$$g(q) = n(m+q)^2 - n^2(q^2+m) = m(m-1)T + n\left\{2ri - (n-1)i^2 - \frac{r^2}{n-1}\right\}, \tag{A.2}$$

where $T = n^2(m-n+1)/\{(n-1)(m-1)\}$. We now consider the various ranges of m that have been used in Theorem 2.1, leading to various cases detailed below.

Case 1: Let $m \in \mathcal{R}_{11}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r < \frac{n}{2}$ and either $i = 0$ or $i = 1$. Also, since $n \equiv 0(\text{mod } 4)$, (A.1) implies that $r \equiv (2-i)(\text{mod } 4)$, $i = 0, 1$. Then, using (A.2), the lower bound $B_{11}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 2n(n-2)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n - 4 - (n-1)i^2 + 2ri - \frac{r^2}{n-1}\right\}. \tag{A.3}$$

For $n \equiv 0(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive p and $0 \leq r < \frac{n}{2}$, substituting $i = 0$ and $i = 1$ in (A.3) gives the respective lower bounds (LB) as

$$LB = T + \frac{n}{m(m-1)}\left\{2n - 4 - \frac{r^2}{n-1}\right\}, r \equiv 2(\text{mod } 4), \quad (A.4)$$

$$LB = T + \frac{n}{m(m-1)}\left\{n + 2r - 3 - \frac{r^2}{n-1}\right\}, r \equiv 1(\text{mod } 4). \quad (A.5)$$

Case 2: Let $m \in \mathcal{R}_{12}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r < \frac{n}{2}$ and $i = -1$. Also, since $n \equiv 0(\text{mod } 4)$, (A.1) implies that $r \equiv 3(\text{mod } 4)$. Then, with $i = -1$, using (A.1) and (A.2), the lower bound $B_{12}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) - 2n(n-2) + 4n|m - q(n-1)|}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}. \quad (A.6)$$

For $n \equiv 0(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive p $(0 \leq r < \frac{n}{2})$, (A.6) gives the LB as

$$LB = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}, r \equiv 3(\text{mod } 4). \quad (A.7)$$

Case 3: Let $m \in \mathcal{R}_{13}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r < \frac{n}{2}$ and $i = 2$. Also, since $n \equiv 0(\text{mod } 4)$, (A.1) implies that $r \equiv 0(\text{mod } 4)$. Then, with $i = 2$, using (A.2), the lower bound $B_{13}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 4n(n-1)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}. \quad (A.8)$$

For $n \equiv 0(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive p $(0 \leq r < \frac{n}{2})$, (A.8) gives the LB as

$$LB = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}, r \equiv 0(\text{mod } 4). \quad (A.9)$$

Case 4: Let $m \in \mathcal{R}_{21}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r < \frac{n}{2}$ and either $i = 0$ or $i = 1$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is even, and equation (A.1) implies that $(q \mp i)n \equiv 2i(\text{mod } 4)$ leading to $r \equiv (2 + i)(\text{mod } 4)$, $i = 0, 1$. Also, $(q \mp i) \equiv i(\text{mod } 2)$, $i = 0, 1$. Then, using (A.2), the lower bound $B_{21}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 2n(n-2) + 8}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n - (n-1)i^2 + 2ri - 4 + \frac{8}{n} - \frac{r^2}{n-1}\right\}. \quad (A.10)$$

For $n \equiv 2(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive $p \equiv i(\text{mod } 2)$ $(0 \leq r < \frac{n}{2})$, substituting $i = 0$ and $i = 1$ in (A.10) gives the respective LB as

$$LB = T + \frac{n}{m(m-1)}\left\{2n - 4 + \frac{8}{n} - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 2(\text{mod } 4), \quad (A.11)$$

$$LB = T + \frac{n}{m(m-1)}\left\{n + \frac{8}{n} + 2r - 3 - \frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 3(\text{mod } 4). \tag{A.12}$$

Case 5: Let $m \in \mathcal{R}_{22}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \le r < \frac{n}{2} - 1$ and $i = -1$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is even, and equation (A.1) implies that $(q \pm 1)n \equiv 2(\text{mod } 4)$ leading to $r \equiv 1(\text{mod } 4)$. Also, $(q \pm 1) \equiv 1(\text{mod } 2)$. Then, with $i = -1$, using (A.1) and (A.2), the lower bound $B_{22}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) - 2n(n-10) + 4(n-2)|m - q(n-1)| - 24}{m(m-1)} = T +$$
$$\frac{n}{m(m-1)}\left\{2r - \frac{8r}{n} + n - \frac{16}{n} + 9 - \frac{r^2}{n-1}\right\}. \tag{A.13}$$

Note that since $n \equiv 2(\text{mod } 4)$ and $r \equiv 1(\text{mod } 4)$, therefore $(\frac{n}{2} - 1) \equiv 0(\text{mod } 2)$ and $r \neq \frac{n}{2} - 1$. Thus in the above range for r, we can take $r < \frac{n}{2}$.

For $n \equiv 2(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive $p \equiv 1(\text{mod } 2)$ $(0 \le r < \frac{n}{2})$, (A.13) gives the LB as

$$LB = T + \frac{n}{m(m-1)}\left\{2r - \frac{8r}{n} + n - \frac{16}{n} + 9 - \frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 1(\text{mod } 4). \tag{A.14}$$

Case 6: Let $m \in \mathcal{R}_{23}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \le r \le \frac{n}{2}$ and $i = 2$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is even, and equation (A.1) implies that $(q \mp 2)n \equiv 0(\text{mod } 4)$ leading to $r \equiv 0(\text{mod } 4)$. Also, $(q \mp 2) \equiv 0(\text{mod } 2)$. Then, with $i = 2$, using (A.2), the lower bound $B_{23}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 4n(n-1)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}. \tag{A.15}$$

Note that since $n \equiv 2(\text{mod } 4)$ and $r \equiv 0(\text{mod } 4)$, therefore $\frac{n}{2} \equiv 1(\text{mod } 2)$ and $r \neq \frac{n}{2}$. Thus in the above range for r, we can take $r < \frac{n}{2}$.

For $n \equiv 2(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive $p \equiv 0(\text{mod } 2)$ $(0 \le r < \frac{n}{2})$, (A.15) gives the LB as

$$LB = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 0(\text{mod } 4). \tag{A.16}$$

Case 7: Let $m \in \mathcal{R}_{31}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \le r < \frac{n}{2}$ and either $i = 0$ or $i = 1$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is odd, and equation (A.1) implies that $(q \mp i)n \equiv 2(1-i)(\text{mod } 4)$ leading to $r \equiv i(\text{mod } 4)$, $i = 0, 1$. Also, $(q \mp i) \equiv (1-i)(\text{mod } 2)$, $i = 0, 1$. Then, using (A.2), the lower bound $B_{31}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 2n(n-2) + x}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n - (n-1)i^2 + 2ri - 4 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \tag{A.17}$$

where $x = 32$ if $m \equiv (1+2j)(\text{mod } 4)$, $(m+q) \equiv (6-4j)(\text{mod } 8)$, for $j = 0$ or $j = 1$, else $x = 0$.

Now, using Lemma 2.1, it follows that $m \equiv (1+2j)(\text{mod } 4)$ if and only if $m + q = m + 4[\frac{m+(1+2j)(n-1)}{4(n-1)}] + 2 - (1+2j) = m - (1+2j) + 4[\frac{m+(1+2j)(n-1)}{4(n-1)}] + 2$. Therefore, $m \equiv (1+2j)(\text{mod } 4)$ and $(m+q) \equiv (6-4j)(\text{mod } 8)$ is equivalent to saying $\left\{m - (1+2j) + 4[\frac{m+(1+2j)(n-1)}{4(n-1)}]\right\} \equiv 4(1-j)(\text{mod } 8)$. Thus, $\left\{\frac{m-(1+2j)}{4} + [\frac{m+(1+2j)(n-1)}{4(n-1)}]\right\} \equiv (1-j)(\text{mod } 2)$, $j = 0, 1$ are the conditions when $x = 32$. Similarly, it can be verified that $\left\{\frac{m-(1+2j)}{4} + [\frac{m+(1+2j)(n-1)}{4(n-1)}]\right\} \equiv j(\text{mod } 2)$, $j = 0, 1$ are the conditions when $x = 0$.

For $n \equiv 2(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive $p \equiv (1-i)(\text{mod } 2)$ $(0 \leq r < \frac{n}{2})$, substituting $i = 0$ and $i = 1$ in (A.17) gives the respective LB as

$$LB = T + \frac{n}{m(m-1)}\left\{2n - 4 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 0(\text{mod } 4), \tag{A.18}$$

$$LB = T + \frac{n}{m(m-1)}\left\{n + 2r - 3 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 1(\text{mod } 4). \tag{A.19}$$

where $x = 32$ if $m \equiv (1+2j)(\text{mod } 4)$ and $\left\{\frac{m-1-2j}{4} + [\frac{m+(1+2j)(n-1)}{4(n-1)}]\right\} \equiv (1-j)(\text{mod } 2)$, $j = 0, 1$, else $x = 0$.

Case 8: Let $m \in \mathcal{R}_{32}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r \leq \frac{n}{2}$ and $i = -1$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is odd, and equation (A.1) implies that $(q \pm 1)n \equiv 0(\text{mod } 4)$ leading to $r \equiv 3(\text{mod } 4)$. Also, $(q \pm 1) \equiv 0(\text{mod } 2)$. Then, with $i = -1$, using (A.1) and (A.2), the lower bound $B_{32}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) - 2n(n-2) + 4n|m - q(n-1)|}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}. \tag{A.20}$$

When $r = \frac{n}{2}$, it can be verified that the expression in (A.20) is the same as the expression that one would get on substituting $r = \frac{n}{2} - 1$ in (A.22). Therefore in the above range for r, we can take $r < \frac{n}{2}$.

For $n \equiv 2(\text{mod } 4)$ and $m = p(n-1) \pm r$ for some positive $p \equiv 0(\text{mod } 2)$ $(0 \leq r < \frac{n}{2})$, (A.20) gives the LB as

$$LB = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 3(\text{mod } 4). \tag{A.21}$$

Case 9: Let $m \in \mathcal{R}_{33}$. Then $m = (q \mp i)(n-1) \pm r$ for some $0 \leq r < \frac{n}{2} - 1$ and $i = 2$. Also, the facts $n \equiv 2(\text{mod } 4)$, q is odd, and equation (A.1) implies that $(q \mp 2)n \equiv 2(\text{mod } 4)$ leading to $r \equiv 2(\text{mod } 4)$. Also, $(q \mp 2) \equiv 1(\text{mod } 2)$. Then, with $i = 2$, using (A.1) and (A.2), the lower bound $B_{23}/\{m(m-1)\}$ in Theorem 2.1 reduces to

$$\frac{g(q) + 4n(n-3) + 8|m - q(n-1)| + 8}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r - \frac{8r}{n} - \frac{8}{n} + 8 - \frac{r^2}{n-1}\right\}. \tag{A.22}$$

As noted in the previous case, when $r = \frac{n}{2} - 1$, the expression in (A.22) is the same as the expression that one would get on substituting $r = \frac{n}{2}$ in (A.20). Therefore in the above range for r, we can take $r < \frac{n}{2}$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv 1 \pmod 2$ $(0 \le r < \frac{n}{2})$, (A.22) gives the LB as

$$LB = T + \frac{n}{m(m-1)} \left\{ 4r - \frac{8r}{n} - \frac{8}{n} + 8 - \frac{r^2}{n-1} \right\}, \quad p \text{ odd}, \quad r \equiv 2 \pmod 4. \qquad (A.23)$$

Summarizing all the above cases, we have the following:
(i) (A.4), (A.5), (A.7) and (A.9) give the lower bounds for $E(s^2)$ when $n \equiv 0 \pmod 4$, $m = p(n-1) \pm r$, p positive and $0 \le r < \frac{n}{2}$,
(ii) (A.11), (A.16), (A.19) and (A.21) give the lower bounds for $E(s^2)$ (subject to the fact that $E(s^2) \ge 4$) when $n \equiv 2 \pmod 4$, $m = p(n-1) \pm r$, p even and $0 \le r < \frac{n}{2}$,
(iii) (A.12), (A.14), (A.18) and (A.23) give the lower bounds for $E(s^2)$ (subject to the fact that $E(s^2) \ge 4$) when $n \equiv 2 \pmod 4$, $m = p(n-1) \pm r$, p odd and $0 \le r < \frac{n}{2}$.

A. 7. Proof of Theorem 4.1.
For $n \equiv 2 \pmod 4$, let H be a Hadamard matrix of order $n+2$ such that, without loss of generality, the first row and first column of H has all $+1$'s. Delete the first row and first column of H and call the resultant $(n+1) \times (n+1)$ matrix G. Now, there are $n/2$ columns of G, each of which has $+1$ in the first row. Let these columns be labelled as $c_1, c_2, \ldots, c_{n/2}$ and let $\mathcal{C} = \{c_1, c_2, \ldots, c_{n/2}\}$, $\mathcal{C}' = \{1, 2, \ldots, n+1\} - \mathcal{C}$. Delete first row of G and call the resultant $n \times (n+1)$ matrix F.

Consider the $n \times n/2$ sub-matrix E consisting of the columns $c_1, c_2, \ldots, c_{n/2}$ of F. Carry out the following operation: In any row of E replace all -1's by $+1$'s. This would affect only certain columns of E where a -1 was replaced by $+1$. Do the same operation for the remaining 'unaffected' columns of E and carry out this process iteratively till there are no unaffected columns in E. Call the resultant matrix $\bar{E}$.

Let X be the $n \times (n+1)$ matrix obtained by replacing E by $\bar{E}$ in F. Let s_{ij} be the (i,j)th element of $X'X$. Then, $s_{ij} = \pm 2$ for all $i \ne j$, since
(i) for $i, j \in \mathcal{C}'$, $s_{ij} = -2$,
(ii) for $i, j \in \mathcal{C}$, $s_{ij} = \pm 2$,
(iii) for $i \in \mathcal{C}$ and $j \in \mathcal{C}'$, $s_{ij} = \pm 2$.
The rest of the proof follows from the fact that for $n \equiv 2 \pmod 4$, $E(s^2) \ge 4$.

A. 8. Proof of Theorem 4.2.
Let Y be the $n \times M$ matrix, whose columns represent factors, such that for any two columns $\boldsymbol{u} = (u_1, \ldots, u_n)'$ and $\boldsymbol{v} = (v_1, \ldots, v_n)'$ of Y, $\boldsymbol{u} \ne \pm \boldsymbol{v}$. Let $s_{uv} = \boldsymbol{u}'\boldsymbol{v}$. The values that s_{uv} can take are $(n-4), (n-8), \ldots, -(n-8), -(n-4)$. Without loss of generality, let $\boldsymbol{u} = (\mathbf{1}_t', -\mathbf{1}_t')'$,

where $t = n/2$ and $\mathbf{1}_t$ is a $t \times 1$ vector of all ones. Without loss of generality, the first element of $\boldsymbol{v}$ can be taken to be 1. Then, $s_{uv} = n - 4$ when exactly one among $v_2, \ldots, v_t$ equals -1 (rest being 1) and exactly one among $v_{t+1}, \ldots, v_n$ equals 1 (rest being -1). Thus, there are $\binom{t-1}{1}\binom{t}{1}$ choices of $\boldsymbol{v}$ for which $s_{uv} = n - 4$. Similarly, there are
$\binom{t-1}{2}\binom{t}{2}$ choices of $\boldsymbol{v}$ for which $s_{uv} = n - 8$,
$\binom{t-1}{3}\binom{t}{3}$ choices of $\boldsymbol{v}$ for which $s_{uv} = n - 12$,
...
$\binom{t-1}{t-2}\binom{t}{t-2}$ choices of $\boldsymbol{v}$ for which $s_{uv} = -(n - 8)$,
$\binom{t-1}{t-1}\binom{t}{t-1}$ choices of $\boldsymbol{v}$ for which $s_{uv} = -(n - 4)$.

The total number of choices of $\boldsymbol{v}$, with $\boldsymbol{u}$ fixed, is

$$\sum_{i=1}^{t-1} \binom{t-1}{i}\binom{t}{i} = \binom{2t-1}{t-1} - 1 = M - 1. \qquad (A.24)$$

From above we see that for $l = 1, \ldots, M$, the sum of squares of the elements of the l-th row (or column) of $Y'Y$ is

$$n^2 + \sum_{i=1}^{t-1} \binom{t-1}{i}\binom{t}{i}(n-4i)^2 = 4\sum_{i=0}^{t-1} \binom{t-1}{i}\binom{t}{i}(t-2i)^2 = \frac{4t^2\binom{2t-1}{t-1}}{2t-1} = \frac{n^2 M}{n-1}. \qquad (A.25)$$

The fact that the sum of squares of the elements in any row of $Y'Y$ is $n^2M/(n-1)$ also follows when one notes that $tr(Y'YY'Y) = tr(YY'YY')$ and YY' has all off-diagonal elements equal to $-M/(n-1)$.

Now, let g' be a design obtained from taking all the $M - m$ columns of Y which are not columns (or -1 times the column) of a given design g involving m factors. Also, let E_g be the value of $E(s^2)$ for the design g. Then, from the structure of $Y'Y$ discussed above, it follows that

$$E_{g'} = \frac{(M-2m)n^2M(n-1)^{-1} + m(n^2 + (m-1)E_g) - (M-m)n^2}{(M-m)(M-m-1)}$$

or,

$$E_{g'} = \frac{n^2(M-2m)(M-n+1)}{(n-1)(M-m)(M-m-1)} + \frac{m(m-1)E_g}{(M-m)(M-m-1)}. \qquad (A.26)$$

In order that g' is $E(s^2)$-optimal, we need to minimize $E_{g'}$ or equivalently E_g should be minimum. Now since d is $E(s^2)$-optimal, thus d' is a $E(s^2)$-optimal supersaturated design.

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