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Construction and use of magic squares and magic rectangles

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Abstract

Magic squares are well-known for its very interesting and entertaining combinatorics. We present a new method for constructing magic squares of odd order. A technique for obtaining two orthogonal latin squares of odd orders is provided. These latin squares are used for constructing the magic squares. The method has been shaped in form of an algorithm that is very convenient for writing a computer program for constructing such squares. The methods have been illustrated through some examples. Finally, some results on magic rectangles is given.

Keywords : Proper magic squares; magic sum; orthogonal latin square; magic rectangles.

1. Introduction

Magic squares and magic rectangles are well-known for its very interesting and entertaining combinatorics. A magic square of side N is an arrangement of N^2 integers in form of a square array of side N such that the sum of its rows, columns and the two diagonals are equal. This sum is called magic sum. These magic squares have been used in designing experiments. For example, Phillips (1964, 1968a, 1968b) illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. As highly balanced structures, magic squares and magic rectangles can be potential tools for use in situations yet unexplored. There is a need to search and investigate on such situations. Magic squares of order N with consecutive and distinct numbers from 0 to N^2-1 is called a proper magic square. It is sometimes a limitation to have such squares and rectangles with consecutive integers starting from 0. To increase the scope of their use it is necessary to allow them to have varying numbers. This would help to apply these magic figures for practical work problems requiring suitable distributions of sets of varying, or even specified, numbers in form of such arrangements. The final aim could be that given a set of numbers how to arrange them in form of a magic square or rectangle.

Variation among a set of numbers arranged in form of magic squares or rectangles can be achieved through a linear transformation. For example, every element, i, of the square/rectangle, can be transformed using the transformation $I = a+bi$ where a and b are suitably chosen constants. As an illustration, if the original magic square or rectangle be written using consecutive number from 0 onwards, and if the transformation uses $a = 1$ and $b = 2$, then all the elements in the arrangement will be only odd numbers starting with 1.

In the present paper we obtain proper magic squares and later use them to construct magic rectangles. The method of construction of magic squares provided here for odd order is extremely simple as it is presented in a ready-to-write form. The magic rectangles have been obtained by suitably selecting the required number of rows from a corresponding magic square. Evidently the elements of the magic rectangle, in this situation, need not be consecutive. An example has also been included to illustrate the type of situations (assignment of workers to work components) where these can have possible use.

2. Construction of orthogonal latin squares and magic squares

Several types of orthogonal latin squares have been suggested in the literature; see for example, Denes and Keedwell (1974). Such orthogonal squares have been used to construct magic squares. We first provide an easy method of construction of orthogonal latin squares of odd orders. Any latin square with its row, column and diagonal sums all equal is called a pseudo magic square. Das and Das (2001) provided a method for constructing magic squares as sums of two pseudo magic squares.

A pseudo magic square of odd order, N, can be obtained by developing a row of N numbers. The numbers $0, 1, 2, \ldots, N-1$ are used to form the first row of a pseudo magic square. By writing the numbers in an arithmetic progression series with its first term as $(N+1)/2$ and 1 as common difference, we get the first row. While obtaining this row, the elements are reduced $\pmod N$. That is to say, when a term is N in the series it is replaced by 0 and the further terms are obtained by increasing from 0.

This way of choosing the first row to obtain the latin square simplifies construction of magic square as it is automatically a pseudo magic square and allows for further automation.

For example, when $N = 9$ the first row starts with $(N+1)/2 = 5$ and is

$$5 \quad 6 \quad 7 \quad 8 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

Developing this row $\pmod 9$ the following pseudo magic square is obtained.

5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3

It is easy to see that it is a pseudo magic square since its diagonal sums are the same as row or/and column sums.

By multiplying the first row of the above square by 2 and reducing $\pmod 9$ the first row of a second pseudo magic square is obtained. By developing this row $\pmod 9$, another pseudo magic square is obtained. These two squares as latin squares are orthogonal. Other such mutually orthogonal squares can also be obtained by multiplying the first row of the first square by x where x is any number from 2 to $N-1$ when N is prime. When N is not a prime, $x=2$ always provides an orthogonal latin square.

We now provide a method for constructing magic squares of odd order using orthogonal latin squares constructed above. Let A denote the matrix of the first latin square and B that of the second latin square. Then a proper magic square, M, of order N is given by $M = NA + B$. Later in this section, we give a simple constructional algorithm the logic of which lies inherent in the above method of construction of magic squares.

As a matter of fact A and B can be any two of the orthogonal pseudo magic squares and any linear function of A and B, like $M = xA + yB$, where one of x and y are each greater than or equal to N, can be used. If $x = N$ and $y = 1$ the corresponding magic square have all numbers from 0 to $N^2 - 1$ as its elements and we denote this magic square by M_0.

As indicated earlier if in M_0 every element i is replaced by $a + bi$ (a and b are positive integers), then the resulting square is also a magic square with its elements in arithmetic progression where a is the first term and b is the common difference. If $a = 1$ and $b = 2$ then the elements of magic square are the odd numbers from 1 to $2N^2 - 1$. Again, if $a = 0$ and $b = 2$ then the elements are the even numbers from 0 to $2N^2 - 2$. In general a is the minimum and $a + b(N^2 - 1)$ is maximum of the elements after such transformation of the elements in M_0. If S is the magic sum in the original magic square, the magic sum after transformation becomes $aN + bS$. These transformation properties hold for magic rectangles also.

Such transformations for the elements in the magic figures generate variability among the elements and this increases the scope of application of magic squares and magic rectangles for solving practical problems as illustrated subsequently.

An algorithm to simplify construction of magic squares of odd orders is now presented.

Writing of the two pseudo magic squares A and B, and then using them to arrive at the magic square can be avoided by following the steps as below:

1. The first element in the top row of the square is taken as $F = N(N+1)/2 + 1$.

2. The top row elements are in arithmetic progression with F as the start term and $N + 2$ as the common difference except that when a term exceeds N^2 it is taken as 0 (only for top row). Alternatively, we can say that in the $(N+1)/2$-th term of the row, the sum exceeds N^2 and is taken as 0.

3. A subsequent row is obtained from its previous row by adding $N + 1$ reduced $\pmod{N^2}$. Furthermore, when a sum if divisible by N then N is subtracted from it.

Continuing with the example of $N = 9$, we illustrate the algorithm by constructing the magic square. We take the first term as $F = 9(10)/2 + 1 = 46$, Common differences $N + 2 = 11$. Also, $N + 1 = 10$.

$$M_0 = \begin{array}{ccccccccc}
46 & 57 & 68 & 79 & 0 & 11 & 22 & 33 & 44 \\
56 & 67 & 78 & 8 & 10 & 21 & 32 & 43 & 45 \\
66 & 77 & 7 & 9 & 20 & 31 & 42 & 53 & 55 \\
76 & 6 & 17 & 19 & 30 & 41 & 52 & 54 & 65 \\
5 & 16 & 18 & 29 & 40 & 51 & 62 & 64 & 75 \\
15 & 26 & 28 & 39 & 50 & 61 & 63 & 74 & 4 \\
25 & 27 & 38 & 49 & 60 & 71 & 73 & 3 & 14 \\
35 & 37 & 48 & 59 & 70 & 72 & 2 & 13 & 24 \\
36 & 47 & 58 & 69 & 80 & 1 & 12 & 23 & 34
\end{array}$$

Before closing this section, we give an illustration of situations for possible use of magic squares.

Consider a contractor who has taken a job that has three components of activities. The job is required to be completed in 3 periods of time, say, day, month or the like. The contractor has fixed a given total number of man hours for the job. These man hours have to be so distributed over the 3 components in different periods that in each of the 3 periods of work equal total numbers of man-hours are used.

We use a 3×3 magic square shown below to provide a distribution of man-hours in each of the 3 periods so as to satisfy the above requirements. The elements in the magic square are the man-hours, the rows denote the job components (JC) and the columns denote the periods (P).

The 3×3 magic square M_0 with 9 elements from 0 to 8 as man-hours is presented below.

Periods	$\mathbf{P_1}$	$\mathbf{P_2}$	$\mathbf{P_3}$	**Total**
$\mathbf{JC_1}$	7	0	5	12
$\mathbf{JC_2}$	2	4	6	12
$\mathbf{JC_3}$	3	8	1	12
Total	12	12	12	36

The total of 36 man-hours are distributed equally over the levels of both the job components and periods. The number 0 in Job comp. 1 and Period 2 indicates that no man hour is employed in this component-period combination. As the restriction for equality of diagonal totals is not there, each of the 3 rows as a whole can be shifted over the components suitably. Similarly, each of the columns as a whole can also be suitably shifted over the periods.

Furthermore, the total man-hours and the individual man-hours can be transformed suitably, as needed, by transforming the elements using the relation $a+bi$ where i stands for the man-hours in the above square and a and b are suitable constants to determine the given total man-hours and also the individual man-hours in the square.

In the above illustration, it is necessary that the number of job components and number of periods are equal. This is a serious limitation from application point of view. This limitation can be removed by using magic rectangles instead. In the next section, we discuss a method for constructing magic rectangles using magic squares of odd orders.

3. Construction of magic rectangles using magic squares

We use the magic squares constructed in Section 2. The method is first introduced using a 7×7 magic square as below.

$$M_0 = \begin{array}{cccccccc} R_1 & 29 & 38 & 47 & 0 & 9 & 18 & 27 \\ R_2 & 37 & 46 & 6 & 8 & 17 & 26 & 28 \\ R_3 & 45 & 5 & 7 & 16 & 25 & 34 & 36 \\ R_4 & 4 & 13 & 15 & 24 & 33 & 35 & 44 \\ R_5 & 12 & 14 & 23 & 32 & 41 & 43 & 3 \\ R_6 & 20 & 22 & 31 & 40 & 42 & 2 & 11 \\ R_7 & 21 & 30 & 39 & 48 & 1 & 10 & 19 \end{array}$$

A magic rectangle with R rows and N columns is an arrangement of RN numbers in a rectangle containing RN cells such that each of the row sums are equal and each of the column sums are also equal. The row sum is equal to the column sum only when $R = N$.

The row sum of the magic square M_0 of order N is $N(N^2-1)/2$. So, the total of the R row sums is $RN(N^2-1)/2$. Hence, the column sums of the magic rectangle of order $R \times N$ is $R(N^2-1)/2$. Thus if $N = 5$ and $R = 2$ the column sum of a magic rectangle of order 2×5 is 24 and its row sum is 60.

We now indicate how to obtain a magic rectangle of order $M \times N$ where M is even and N is odd. To obtain a magic rectangle with 2 rows and N columns the first row, R_1 of M_0 is written as such. The second row is the last row of M_0 (i.e., R_7) written in reverse order. Thus, using M_0 of order 7×7 as presented above we get the 2×7 magic rectangle as below.

2×7 magic rectangle

$$\begin{array}{ccccccc} 29 & 38 & 47 & 0 & 9 & 18 & 27 \\ 19 & 10 & 1 & 48 & 39 & 30 & 21 \end{array}$$

It is seen that the row sum of above magic rectangle is 168 and the column sum is 48.

Using R_2 and R_6 similarly, we get another 2×7 magic rectangle. A third magic rectangle of order 2×7 is obtained using R_3 and R_5 . Taking together any two of the above mentioned three 2×7 magic rectangles we get a magic rectangle of order 4×7 . Again, taking all the above three magic rectangles, each of order 2×7 , we get a magic rectangle of order 6×7 .

Next, we show how to obtain a magic rectangle of order $M \times N$ where M and N are both odd. We start with a magic rectangle of order 3×7 . The column sum of a magic rectangle of order 3×7 is 72. First, select two rows of M_0 such that the sum of two elements in any column of these two rows when subtracted from the column sum, 72, lies between 0 and 48 , both inclusive. Accordingly, we cannot use R_1 and R_2 , nor R_1 and R_3 since these two choices do not satisfy the above conditions. The rows R_1 and R_4 does satisfy the above conditions and so we take these two rows as the first two of the three rows of the magic rectangle. The elements in the third row can now be obtained easily as the difference between 72 and the sum of the corresponding entries in the two rows. It can be verified that the numbers obtained for the third row belong to the last row, R_7 (in rearranged fashion), and as such none of these numbers occur in the first two rows. Hence all these 21 numbers in 3 rows are distinct lying between 0 and 48.

The 3×7 magic rectangle obtained using the above technique is:

R_1	29	38	47	0	9	18	27
R_4	4	13	15	24	33	35	44
R_7	39	21	10	48	30	19	1

We can find other such 3×7 magic rectangle. For example,

R_2	37	46	6	8	17	26	28
R_6	20	22	31	40	42	2	11
R_4	15	4	35	24	13	44	33

Actually, the third row in all $3 \times N$ magic rectangles can be written systematically by first finding the entry (say, z) corresponding to column 1 in third row. Next, in M_0, the row with z is searched out. Let this row be R. The subsequent entries in the third row of the magic rectangle would come from R. For the construction of the above rectangle, one would see that 15 is in column 3 of R_4. After z has been obtained, the next entry for the third row of the magic rectangle is the number left of z in R leaving one column. Proceeding in this way all the entries of the third row are obtained. Here, when the left end of R is reached, we move cyclically to the right of R and proceed as earlier.

The range of the numbers involved in different magic rectangles are different. This property can be exploited to meet requirements in specific applications.

Joining magic rectangles with even number of rows and the one with 3 rows, one can obtain magic rectangles with any odd number of rows. The above method holds for constructing any $M \times N$ magic rectangle for N odd.

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