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To Kalyan Sinha on his sixtieth birthday

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Abstract. We study various conditions on matrices B and C under which they can be the off-diagonal blocks of a partitioned normal matrix.

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The structure of general normal matrices is far more complicated than that of two special kinds — hermitian and unitary. There are many interesting theorems for hermitian and unitary matrices whose extensions to arbitrary normal matrices have proved to be extremely recalcitrant. See, e.g., [1]. The problem whose study we initiate in this note is another one of this sort.

We consider normal matrices N of size $2n$, partitioned into blocks of size n as

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

Normality imposes some restrictions on the blocks. One such restriction is the equality

$$||B||_2 = ||C||_2 \tag{2}$$

between the *Hilbert-Schmidt (Frobenius) norms* of the off-diagonal blocks B and C. If T is any $m \times m$ matrix with entries t_{ij}, then

$$\|T\|_2 = \left(\sum_{j=1}^{m} |t_{ij}|^2 \right)^{1/2}.$$

The equality (2) is a consequence of the fact that the Euclidean norm of the j th column of a normal matrix is equal to the Euclidean norm of its j th row.

Replacing the Hilbert-Schmidt norm by another unitarily invariant norm, we may ask whether the equality (2) is replaced by interesting inequalities. Let $s_1(T) \geq \cdots \geq s_m(T)$ be the singular values of T. Every unitarily invariant norm $|||T|||$ is a symmetric gauge function of $\{s_j(T)\}$. See [1, Chapter IV] for properties of such norms. Much of our concern in this note is with the special norms

$$\|T\|_2 = (\operatorname{tr} T^*T)^{1/2} = \left(\sum_{j=1}^{m} s_j^2(T)\right)^{1/2},$$

and

$$\|T\| = s_1(T) = \sup_{x\in\mathbb{C}^m, \|x\|=1} \|Tx\|. \tag{3}$$

The latter is the norm of T as a linear operator on the Euclidean space $\mathbb{C}^m$. Clearly

$$\|T\| \leq \|T\|_2 \leq \sqrt{m}\|T\|, \tag{4}$$

for every $m \times m$ matrix T.

If the matrix N in (1) is hermitian, then $C = B^*$, and hence, $|||C||| = |||B|||$ for all unitarily invariant norms. If N is unitary, then $AA^* + BB^* = A^*A + C^*C = I$. Hence, the eigenvalues λ_j satisfy the relations

$$\begin{aligned} \lambda_j(BB^*) &= \lambda_j(I - AA^*) = 1 - \lambda_j(AA^*) \\ &= 1 - \lambda_j(A^*A) = \lambda_j(I - A^*A) = \lambda_j(C^*C). \end{aligned}$$

Thus B and C have the same singular values, and again $|||B||| = |||C|||$ for all unitarily invariant norms.

This equality of norms does not persist when we go to arbitrary normal matrices, as we will soon see. From (2) and (4) we get a simple inequality

$$||B|| \leq \sqrt{n}\ ||C||. \tag{5}$$

One may ask whether the two sides of (5) can be equal, and that is the first issue addressed in this note.

When $n = 2$, it is not too difficult to construct a normal matrix N of the form (1) in which $\|B\| = \sqrt{2}\|C\|$. One example of such a matrix is

$$N = \left[\begin{array}{cc|cc} 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right]. \tag{6}$$

When $n = 3$, examples seem harder to come by. One that preserves some of the features of (6) is given by the matrix

$$N = \left[\begin{array}{ccc|ccc} 0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{\sqrt{3}} + 1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}} + 1} \\ 0 & 1 & 0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 \end{array}\right]. \tag{7}$$

It can be seen that N is normal and plainly $\|B\| = \sqrt{3}$ while $\|C\| = 1$. When $n = 4$ it is impossible to find such a matrix, and that is our first theorem.

The following elementary lemma (which can be verified by induction on the integer k) is used repeatedly in our proof.

Lemma. Let V be an n-dimensional vector space and let $V_1, \ldots, V_k$ be subspaces of V the sum of whose dimensions is larger than $(k-1)n$; i.e.,

$$\sum_{j=1}^{k} \dim V_j > (k-1)n.$$

Then the intersection of these k subspaces is nonzero.

Theorem 1. There exists a normal matrix N of the form (1) with

$$||B|| = \sqrt{n}\ ||C|| \tag{8}$$

if and only if $n \leq 3$.

Proof. Note first that if equalities (2) and (8) hold simultaneously, then rank B must be one and C must be unitary. So, after applying a unitary similarity by $\begin{bmatrix} C & O \\ O & I \end{bmatrix}$, we may assume that

$$N = \begin{bmatrix} A & B \\ I & D \end{bmatrix}. \tag{9}$$

The normality condition $N^*N = NN^*$ leads to two equations

$$A - D = A^*B - BD^*, \tag{10}$$

$$2I = AA^* - A^*A + BB^* + B^*B + D^*D - DD^*. \tag{11}$$

Since B is of rank one

$$\dim(\ker\ B) = \dim(\ker\ B^*) = n - 1,$$

where $\dim X$ stands for the dimension of a space X. So, if $n \geq 3$, then the dimensions of $\ker B$ and $\ker B^*$ add up to more than n. Hence their intersection is nonzero, and we may choose a unit vector x in this intersection. For this vector, we obtain from (10)

$$(A - D)x = -BD^*x, \tag{12}$$

$$(A - D)^*x = B^*Ax. \tag{13}$$

The equation (11) leads to the condition

$$2 = \|A^*x\|^2 - \|Ax\|^2 + \|Dx\|^2 - \|D^*x\|^2. \tag{14}$$

The rest of the proof consists of showing that if $n > 3$, then we can choose a vector $x \in (\ker B) \cap (\ker B^*)$ for which these conditions cannot be satisfied.

The two matrices BD^* and B^*A have rank at most 1, so their kernels have dimension at least $n - 1$. Hence

$$\dim(\ker B) + \dim(\ker B^*) + \dim(\ker BD^*) + \dim(\ker B^*A) \geq 4n - 4. \tag{15}$$

This is larger than $3n$ whenever $n > 4$. So, in this case the four kernel spaces involved in (15) have a nonzero intersection. Let x be a unit vector in this intersection. Then from (12) and (13) we find that

$$(A - D)x = 0 \quad \text{and} \quad (A - D)^*x = 0.$$

Hence, $\|Ax\| = \|Dx\|$ and $\|A^*x\| = \|D^*x\|$. This contradicts the condition (14).

Now consider the case $n = 4$. The spaces $\ker B$ and $\ker B^*$ have dimension 3 each, while the space $\ker B(A+D)^*$ has dimension at least 3. The three dimensions add up to more than 8. Hence, we can find a unit vector x in the intersection of these three spaces. For this vector we have

$$\begin{aligned} \|A^*x\|^2 - \|D^*x\|^2 &= \mathrm{Re}\,\langle (A+D)^*x, (A-D)^*x\rangle \\ &= \mathrm{Re}\,\langle (A+D)^*x, B^*Ax\rangle \\ &= \mathrm{Re}\,\langle B(A+D)^*x, Ax\rangle \\ &= 0. \end{aligned} \tag{16}$$

Here the second equality is a consequence of (13), and at the last step we have used the fact that $B(A+D)^*x = 0$.

Using (12) instead of (13) we get

$$\begin{aligned} \|Dx\|^2 - \|Ax\|^2 &= \mathrm{Re}\,\langle (A+D)x, (D-A)x\rangle \\ &= \mathrm{Re}\,\langle (A+D)x, BD^*x\rangle \\ &= \mathrm{Re}\,\langle B^*(A+D)x, D^*x\rangle. \end{aligned} \tag{17}$$

Since B is a matrix with rank equal to 1 and norm equal to 2, we have $B^*BB^* = 4B^*$. (Use the polar decomposition $B = UP$. In some orthonormal basis P is diagonal with only one nonzero entry 2 on the diagonal. So $B^*BB^* = P^3U^* = 4PU^* = 4B^*$.) Hence we have

$$\begin{aligned} 4B^*Ax &= B^*BB^*Ax \\ &= B^*B(A-D)^*x \quad (\text{using (13)}) \\ &= B^*B(A+D)^*x - 2B^*BD^*x \\ &= -2B^*BD^*x \\ &= 2B^*(A-D)x \quad (\text{using (12)}) \\ &= 4B^*Ax - 2B^*(A+D)x. \end{aligned}$$

This shows that $B^*(A+D)x = 0$, and we get from (17)

$$\|Dx\|^2 - \|Ax\|^2 = 0. \tag{18}$$

Clearly the relations (14), (16) and (18) cannot be simultaneously true.

We have shown that when $n \geq 4$, there cannot exist a $2n \times 2n$ normal matrix of the form (9) in which B is an $n \times n$ matrix of rank one. This proves the theorem. ■

Our discussion leads to some natural questions.

Problem 1. For $n \geq 4$ find the numbers

$$\alpha_n = \sup \left\{ ||B||/||C|| : \exists A, D \text{ for which } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ is normal} \right\}.$$

We have seen $\alpha_n < \sqrt{n}$ for $n \geq 4$. It would be of interest to know whether α_n is a bounded sequence.

Problem 2. What matrix pairs B, C can be the off-diagonal entries of a normal matrix N as in (1)? In other words when does $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ have a normal completion?

Example 1. Consider the 2×2 matrices

$$B = \begin{bmatrix} 1 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

Then, $\|B\|_2 = \|C\|_2$. However, there do not exist any 2×2 matrices A and D for which $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is normal. We leave the verification of this statement to the reader. Thus the equality (2) is only a necessary condition for normality of the matrix (1).

We consider some special cases of the question raised in Problem 2. We assume either $B = C$, or $B = C^*$.

For every B, the matrix $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$ has a normal completion, and this completion may be chosen to be of the special type $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$. Indeed, if U is the unitary matrix $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, then

$$U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}.$$

So $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is normal if and only if $\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}$ is normal, and this is the case if and only if $A+B$ and $A-B$ both are normal. The most obvious choice of A that assures

this is $A = B^*$. Thus

$$\widetilde{B} = \begin{bmatrix} B^* & B \\ B & B^* \end{bmatrix} \tag{19}$$

is a normal completion of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. We have the norm inequality

$$\|B\| \leq \|\widetilde{B}\| \leq 2\|B\|. \tag{20}$$

When $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\|\widetilde{B}\| = \|B\|$. On the other hand if B is any hermitian matrix, then $\|\widetilde{B}\| = 2\|B\|$. In this case, and more generally when B is normal, $\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$ is normal and has norm equal to $\|B\|$. This raises the question of finding completions of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$ that are "optimal" in various senses.

Problem 3. Given a matrix B find a matrix A such that

$$N = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is normal and has least possible norm. This is equivalent to asking for a matrix A such that $A + B$ and $A - B$ are normal and the quantity $\max(\|A + B\|, \|A - B\|)$ is minimised. It might be difficult to find *all* solutions to this problem. The following considerations lead to *one* solution.

We assume that B is a contraction, i.e. $\|B\| \leq 1$ and ask for an A so that $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is unitary. This is a unitary completion of the matrix $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Let $B = USV$ be the singular value decomposition of B. Then

$$\begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} U^*AV^* & S \\ S & U^*AV^* \end{bmatrix}.$$

So, the problem reduces to finding an A' such that $\begin{bmatrix} A' & S \\ S & A' \end{bmatrix}$ is unitary. A familiar idea from the theory of unitary dilations [2, p. 232] suggests the choice $A' = i\left(I - S^2\right)^{1/2}$.

This tells us how to find for any matrix B one of the least-norm normal completions of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Assume $\|B\| = 1$ and find a unitary completion as proposed above.

Next we consider the case $B = C^*$, and ask for matrices A and D such that

$$N = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \tag{21}$$

is normal. A calculation shows that the matrices A and D must be normal and satisfy the equation

$$(A - A^*)\, B = B\, (D - D^*)\,. \tag{22}$$

Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$ be the Cartesian decompositions of A and D. Here (H_1, K_1) and (H_2, K_2) are two pairs of commuting hermitian matrices. The equation (22) is equivalent to $K_1 B = BK_2$. This shows

$$B^*BK_2 = B^*K_1B = (K_1B)^*B = (BK_2)^*B = K_2B^*B.$$

So K_2 commutes with B^*B, and hence with the factor P in the polar decomposition $B = UP$.

Thus the general solution to (22) is obtained as follows. Choose K_0 and K_2, both hermitian, satisfying the conditions

$$K_0P = PK_0,\ \ K_2P = PK_2,\ \ (K_0 - K_2)P = 0.$$

Let $K_1 = UK_0U^*$. This condition ensures

$$K_1B = UK_0U^*B = UK_0P = UK_2P = UPK_2 = BK_2.$$

Choose hermitian matrices H_1 and H_2 that commute with K_1 and K_2, respectively. Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$. This leads to N in (21) being normal.

As before, we consider also the special case $\|B\| \le 1$ and ask for A and D such that the matrix (21) is unitary. This can be solved as follows. Let $B = UP$ be any polar decomposition. Choose hermitian matrices K_0 and K_2 that commute with P and satisfy the inequalities

$$K_0^2 \le I - P^2, \quad K_2^2 \le I - P^2.$$

Then choose hermitian matrices H_0 and H_2 that commute with K_0 and K_2, respectively, and satisfy the conditions

$$H_0^2 + K_0^2 = H_2^2 + K_2^2 = I - P^2.$$

Let $A = U(H_0 + iK_0)U^*$ and $D = H_2 + iK_2$. Then the matrix (21) is unitary.

Example 1 shows that the equality $\|B\|_2 = \|C\|_2$ is not a sufficient condition for the existence of a normal completion of $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$.

Our next proposition shows that equality between all unitarily invariant norms is a sufficient condition.

Proposition. Let B, C be $n \times n$ matrices with $|||B||| = |||C|||$ for every unitarily invariant norm. Then the matrix $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a completion that is a scalar multiple of a unitary matrix.

Proof. If $|||B||| = |||C|||$ for every unitarily invariant norm, then $s_j(B) = s_j(C)$ for all $j = 1, 2, \ldots, n$. Hence, there exist unitary matrices U_1, U_2, V_1, V_2 such that $B = U_1 S U_2$, and $C = V_1 S V_2$. Divide B and C by $||S||$, and thus assume $||S|| = 1$. Then $I - S^2$ is positive, and has a positive square root. It is easy to see that the matrix

$$\begin{bmatrix} (I - S^2)^{\frac{1}{2}} & S \\ S & -(I - S^2)^{\frac{1}{2}} \end{bmatrix}$$

is unitary. Multiply this matrix on the left by the unitary matrix $U_1 \oplus V_1$, and on the right by the unitary matrix $V_2 \oplus U_2$. This gives a unitary matrix whose off-diagonal blocks are B and C. ∎

While the condition in the Proposition is not necessary, it is sensitive to small perturbations. The matrices B and C in Example 1 satisfy the conditions $\|B\|_2 = \|C\|_2$, $|||B||| = |||C||| + O(\varepsilon)$, but for $\varepsilon \neq 0$, there is no possible normal completion of $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$.

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