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Supmech: the Geometro-statistical Formalism Underlying Quantum Mechanics

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A scheme of mechanics, called 'supmech', is developed which aims at providing a base for a solution of Hilbert's sixth problem (seeking a unified axiomatization of physics and probability theory) and serves to develop quantum mechanics autonomously (i.e. without having to *quantize* classical systems). Integrating noncommutative symplectic geometry and noncommutative probability in an algebraic setting, it associates, with every 'experimentally accessible' system, a symplectic superalgebra and operates essentially as noncommutative Hamiltonian mechanics with an extra condition of 'compatible completeness' between observables and pure states incorporated. A noncommutative analogue of the Poincaré-Cartan form is introduced. It is shown that interactions between systems can be consistently described in supmech only if either (i) all system algebras are supercommutative, or (ii) all system algebras are non-supercommutative and have a quantum symplectic structure characterized by a *universal* Planck type constant of the dimension of action. 'Standard quantum systems', defined algebraically, are shown to have faithful Hilbert space - based realizations; the rigged Hilbert space - based Dirac bra-ket formalism naturally appears. The formalism has a natural place for commutative superselection rules. Treating massive particles as localizable elementary quantum systems, the Schrödinger equation for them is obtained without ever using a classical Hamiltonian or Lagrangian. Quantum measurements are satisfactorily treated; the unwanted macroscopic superpositions are shown to be suppressed when the observations on the apparatus are restricted to macroscopically distinguishable pointer readings. This treatment automatically incorporates the decohering effects of the internal environment of the apparatus; a trivial extension also serves to include the external environment.

Contents

Underlying everything you see
There is motion
Governed by
Noncommutative symplectics.

I. INTRODUCTION

The statement of the famous sixth problem of Hilbert [1] (henceforth referred to as H6) reads :

"To treat in the same manner, by means of axioms, the physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics."

It appears reasonable to have a somewhat augmented version of H6; the following formal statement is being hereby proposed for this :

"To evolve an axiomatic scheme covering all physics including the probabilistic framework employed for the treatment of statistical aspects of physical phenomena."

A solution of this problem must include a satisfactory treatment of the dynamics of the universe and its subsystems. Since all physics is essentially mechanics, the formalism underlying such a solution must be an elaborate scheme of mechanics (with elements of probability incorporated). Keeping in view the presently understood place of quantum mechanics (QM) in the description of nature, such a scheme of mechanics must incorporate, at least as a subdiscipline or in some approximation, an ambiguity-free development of QM without resorting to the prevalent practice of *quantization* of classical systems. Since a large class of systems in nature admit a classical description to a very high degree of accuracy, the envisaged mechanics must also facilitate a transparent treatment of quantum-classical correspondence. For this to be feasible, the underlying framework must be such that both quantum and classical mechanics can be described in it [2].

Wightman's article [3] is a decent review of the work relating to the solution of H6 upto mid-seventies. It covers Hilbert's own work in this connection and the main developments relating to the axiomatization of QM and of quantum field theory (QFT). Highlights in his treatment of axiomatization of QM are: von Neumann's Hilbert space based axiomatics; Algebraic generalization of QM by Jordan, von Neumann and Wigner; Segal's postulates for general quantum mechanics; Variants of the quantum logic approach by Birkhoff, von Neumann, Mackey, Piron and others; Hidden variable theories; EPR paradox and the question of completeness of QM. The highlights in the axiomatization of QFT are : Perturbative S-matrix and renormalization; Fock space; Reduction formulas; Representations of the inhomogeneous Lorentz group and relativistic wave equations; Haag' theorem; Wightman formalism; PCT and Spin-statistics theorems in axiomatic field theory; Haag-Ruelle collision theory; C^*-algebraic approach to quantum field theory and statistical mechanics and Constructive field theory. De-

tailed references may be found in Wightman's paper.

Highlights of the developments during the past four decades are : Progress in gauge theories of fundamental interactions [4]; String theory [5,6]; Loop quantum gravity (LQG) [7,8]; Progress in algebraic QFT [9-13] and in Constructive field theory [14,15] and some developments relating to the quantum mechanical formalism [16-27]. None of these appears to provide key ingredients to the solution of H6. In gauge models, concrete progress has been made by the usual not-quite-rigorous field theoretic techniques and they need to be brought within the domain of rigorous mathematics for axiomatization. String theory and LQG are both quantization programs and have yet to develop as concrete autonomous theories. Algebraic QFT has yet to make firm contact with mainstream particle physics. Constructive field theory has yet to show concrete progress in four space-time dimensions. Developments in QM have generally been concerned with the measurement problem, histories approach, quantum nonlocality and quantum information and operational approach to QM; none of these can be claimed to have contributed substantially to the solution of H6.

It appears fair to say that, at present, there does not exist a formalism which can provide for an unambiguous autonomous development of QM which, starting with some appealing basics, connects smoothly to the traditional Hilbert space QM and facilitates a satisfactory treatment of measurements.

In this paper, we shall present such a formalism. It satisfies the somewhat stringent criteria (for an autonomous development of QM) laid down in Ref.[2], namely

(i) the basic concepts and equations of QM should be developed autonomously;
(ii) there should be a framework which can accommodate both QM and CM (classical mechanics); in this framework, the basic concepts and equations of CM should be derivable from those of QM in an appropriate limit.

The traditional development of special relativity, for example, satisfies analogous requirements [2]; there is a transparent $c \to \infty$ limit to Galilean relativity.

[The insistence on an autonomous development of quantum physics is not just a matter of aesthetic satisfaction. It is the author's view that, if we have such a formalism, many problems of theoretical physics, when reformulated in the autonomous quantum framework, will be easier to solve. To get a feel for this, suppose that, instead of having the elaborate (Poincaré group based) formalism for special relativity, we had only somehow found some working rules to *relativize* nonrelativistic equations. The several important results obtained by using Lorentz-covariant formalisms (for example, the covariant renormalization programme of QFT and the theorems of axiomatic QFT) would either have not been obtained or obtained using a highly cumbersome formalism.]

The natural choice for the underlying geometry of the desired formalism is noncommutative geometry (NCG) [28-32]. Noncommutativity is the hallmark of QM. Indeed, the central point made in Heisenberg's paper [33] that marked the birth of QM was that the kinematics

underlying QM must be based on a non-commutative algebra of observables. This idea was developed into a scheme of mechanics — called matrix mechanics — by Born, Jordan, Dirac and Heisenberg [34-36]. The proper geometrical framework for the construction of the quantum Poisson brackets of matrix mechanics is provided by non-commutative symplectic geometry [29,37-39]. The NCG scheme employed in these works is a straightforward generalization of the scheme of commutative differential geometry in which the algebra $C^\infty(M)$ of smooth functions on a manifold M is replaced by a general (not necessarily commutative) complex associative *-algebra $\mathcal{A}$ and the Lie algebra of smooth vector fields on M by that of derivations on $\mathcal{A}$.

While Heisenberg presented the quantum view of observables, Schrödinger [40] dealt with wave functions which, through the Born interpretation [41], brought out the important aspect of QM as an intrinsically probabilistic theory. Noncommutativity of the algebra of observables has important implications relating to the basic character of the operative probability theory – the so-called 'quantum probability' of which a variety of versions/formulations have appeared in literature [42-50]. Among these formulations, the one that suits our needs best is the one provided by the observable-state framework based on complex associative, unital (topological) *-algebras [50-53]. This choice serves the important purpose of allowing us to adopt the strategy of combining elements of noncommutative symplectic geometry and noncommutative probability in an algebraic framework.

The scheme based on normed algebras [43,44,54-56,9-11,57,58], although it makes use of observables and states, does not serve our needs because it is not suitable for the treatment of noncommutative symplectic geometry. Iguri and Castagnino [59] have analyzed the prospects of a more general class of algebras (nuclear, barreled b*-algebras) as a mathematical framework for the formulation of quantum principles better than that of the normed algebras. These algebras accommodate unbounded observables at the abstract level. Following essentially the 'footsteps' of Segal [44], they obtain results parallel to those in the C*-algebra theory — an extremal decomposition theorem for states, a functional representation theorem for commutative subalgebras of observables and an extension of the classical GNS theorem. In a sense, this work is complementary to the present one where the emphasis is on the development of noncommutative Hamiltonian mechanics. We have employed locally convex (super-)algebras restricted by a condition of 'compatible completeness' on the collections of observables and pure states (it is satisfied in classical Hamiltonian mechanics and the traditional Hilbert space QM) which plays a crucial role in connecting the algebraic scheme of mechanics (supmech) to the traditional Hilbert space QM.

In section 4 of Ref[2], a scheme of mechanics based on noncommutative symplectic geometry was introduced; it was designed to provide a proper geometrical setting for the matrix mechanics mentioned above. States were, however, not introduced in the algebraic setting. This work, therefore, falls short of a proper realization of the strategy mentioned above. In the present work, this deficiency has been removed and a proper integration of noncommutative symplectic geometry and noncommutative probability has been achieved. The improvement in the definition of noncommutative differential forms introduced in Ref.[38] [i.e. demanding

$\omega(...,KX,...) = K\omega(...,X,...)$ where K is in the center of the algebra; for notation, see section III] is also incorporated. Moreover, to accommodate fermionic objects on an equal footing with the bosonic ones, the scheme developed here is based on superalgebras. The scheme of mechanics developed along the above lines is given, for easy reference, the name 'Supmech'(short form for 'supersymplectic mechanics').

Supmech has quantum and classical mechanics as special subdisciplines. This fact appears to open the prospects of a consistent treatment of the interaction of a quantum and a classical system. In Ref.[60], the author applied such a formalism to the treatment of measurements in QM providing what appeared to be the most natural solution to the measurement problem in QM. An important ingredient in this work was the Poisson bracket on the tensor product of two algebras [the non-super version of the formula (154) below]. Shortly after that paper appeared in the archive, M.J.W. Hall [61] pointed out to the author that the 'Poisson bracket' mentioned above does not satisfy the Jacobi identity in some cases (as shown, for example, in Ref.[62]). A revised calculation by the author produced results which were partly discouraging [in that a 'natural'/'canonical' symplectic structure on the tensor product of a (super-)commutative and a non- (super-)commutative (super-)algebra (both the (super-)algebras being of the above mentioned type) does not exist] and partly very very interesting : a symplectic structure of the above sort on the tensor product of two non-(super)commutative (super-)algebras exists if and only if each of the (super-)algebras has a 'quantum symplectic structure' [i.e. one which gives a Poisson bracket which is a (super-)commutator up to multiplication by a constant $(i\lambda^{-1})$ where λ is a real-valued constant of the dimension of action] characterized by a *universal* parameter λ. The formalism, therefore, has a natural place for the Planck constant as a universal constant — just as special relativity has a natural place for a universal speed. In fact, the situation in supmech is somewhat better because, whereas, in special relativity, the existence of a universal speed is *postulated*, in supmech, the existence of a universal Planck-like constant is *dictated/predicted* by the formalism.

The negative result about the possibility of a consistent quantum-classical interaction in the supmech framework is by no means 'fatal' for the treatment of measurement problem in supmech. It turns out that it is adequate to treat the apparatus as a quantum system approximated well by a classical system (in the sense of, for example, phase space descriptions of quantum and classical dynamics). It is shown in section VIII that such a treatment reproduces the results of Ref.[60].

The detailed plan of the rest of the paper is as given in the contents. In section II, we present arguments, based on physics fundamentals, for adopting the kind of formalism that we do. These arguments make it quite plausible that the formalism being evolved is the appropriate one for doing physics at the deepest level. In section III, essential developments in the (super-) derivation -based noncommutative differential calculus and symplectic structure are presented. The induced mappings on (super-)derivations and differential forms (Φ_* and Φ^*— analogues of the push-forward and pull-back mappings induced by diffeomorphisms in the traditional differential geometry) are described; they play important roles in supmech. In section IV,

the formalism of supmech is developed. A special feature of this formalism is the condition of 'compatible completeness' mentioned above. The development includes, besides the basic formalism as noncommutative Hamiltonian mechanics, a treatment of symplectic actions of Lie groups and the noncommutative avatars of the Poincaré- Cartan form and the symplectic version of Noether's theorem. A general treatment of localizable systems is also given. In section V, elementary systems are defined in supmech and the special cases of nonrelativistic and relativistic elementary systems are treated. The role of relativity groups in the identification of fundamental observables of elementary systems is emphasized. Particles are treated as localizable elementary systems. In section VI, coupled systems are treated and the results about the symplectic structure on the tensor products of the superalgebras mentioned above are obtained. (A reasonably self-contained presentation of the non-super version of these results was given in Ref.[63].)

Section VII is devoted to the treatment of quantum and classical systems as special categories of systems in supmech. Quantum systems are taken up before classical systems to emphasize the autonomous nature of the treatment of QM. 'Standard quantum systems' are defined in the algebraic setting; the CC condition ensures the existence of their Hilbert space - based faithful realizations.The formalism is shown to have a natural place for commutative superselection rules. A transparent treatment of quantum - classical correspondence is given emphasizing some formal aspects. The superclassical extension of classical mechanics (incorporating fermionic objects in the setting of a supercommutative superalgebra) is treated and is shown (for the case of a finite number of fermion generators) to generally violate the CC condition which disqualifies it from being a bonafide subdiscipline of supmech. In section VIII, measurements in quantum systems are treated and, a straightforward solution of the measurement problem is given along the lines mentioned above; the treatment of the apparatus (properly as a system) automatically incorporates the desirable decoherence effects to suppress the unwanted macroscopic quantum interference terms. In section IX, a set of axioms underlying the work presented in sections IV-VIII is given. The last section contains some concluding remarks.

II. FROM BASICS TO ALGEBRAS

In this section, we shall present arguments based on fundamentals relating to physical theories, for making the choice of the type of formalism for supmech.

We look for the ingredients that should go into the formalism that is intended to cover *all physics*. To this end, we start by considering the primitive elements which every physical theory — classical, quantum, or more general — is expected to possess (explicitly or implicitly). The author came across the term 'primitive elements of physical theory' (PEPT) in a not so well known but an instructive and insightful paper by Houtappel, Van Dam and Wigner (HVW) [64] which aimed at a treatment of symmetries [especially the 'geometric invariance principles' (space-time symmetries)] in a very general setting involving the PEPT which, according to

HVW, were 'measurements and their results'. We, however, would like to have a reasonably 'complete' minimal set of such primitive elements. (The word 'complete' here should be understood in the intuitive sense of 'good enough for doing some concrete physics'.) To this end, a promising route to take is to have a close look at the ingredients going into the construction of the mathematical objects employed by HVW (the Π-functions — the 'forefathers' of the objects presently known as histories [65,66,23,67]). Doing this, one finds that such a minimal set may be taken as [68, last section]

(i) observations/measurements;
(ii) description of evolution of systems (typically in terms of a discrete or continuous parameter called 'time');
(iii) conditional predictions about systems : given some information about a system (typically in terms of values of appropriate measurable quantities at one or more instants of time), to make predictions/retrodictions about its behavior.

Item (iii) above generally involves elements of probability. It is not difficult to see why Hilbert, while formulating his VIth problem, chose to put probability along with physics. Any formal axiomatization of probability must include hypotheses about the way uncontrollable random influences affect outcomes of experiments. Since these influences have physical causes, any comprehensive (theory construction)/axiomatization of physics must appropriately treat these causes. In any physical theory, the theoretical apparatus employed to cover item (iii) above is very much a part of the theory and, in case a (partial or total) failure of the theory occurs, may well have to be subjected to scrutiny along with other ingredients of the theory.

A useful concept that serves to introduce elements of probability in a sufficiently general way and integrate items (i)-(iii) above is that of *state*. A state of a system encodes available information about the system in terms of values of appropriate *observables* (i.e. measurable quantities). Evolution of systems can be described in terms of change of state with time. Problems of conditional predictions can be formulated in terms of probabilities of transitions of systems prepared in given states to various possible states.

This reasoning leads to the prospects of a reasonably economical and general description of systems in terms of observables and states. In contrast to, for example, Araki's book [11], where these objects were introduced through analysis of experiments, we have introduced them by considering some basics of theory construction. This is obviously more in tune with the general theme of the present work.

The traditional algebraic schemes (generally based on C*-algebras) have employed these objects as basic structures and have achieved some good results. They, however, do not realize the true potential of such an approach. In these works, one generally puts the Weyl form of commutation relations (for finite as well infinite number of degrees of freedom) 'by hand' without connecting them to some underlying geometry. This deficiency can be overcome by dropping the restriction to normed algebras. As pointed out in the previous section, the underlying geometry of QM is noncommutative symplectic geometry whose vehicles are complex associative

algebras. A program based on such algebras, integrating noncommutative symplectic geometry with noncommutative probability (arrived at in the previous section as an intuitively appealing step taken after noting the two roles of these algebras), therefore, appears to be the best bet for the desired formalism also from considerations based on physics fundamentals.

Taking an appropriate class of (super-)algebras as the basic objects, we shall define noncommutative symplectic structure on them. The promised mechanics (Supmech) will be developed in the form of Hamiltonian mechanics in the setting of this structure. We shall try to exploit the underlying noncommutative symplectic geometry as much as possible . For systems admitting a space-time description, for example, we shall insist on the action of the relativity group on the system algebra to be a Poisson action (for its definition, see section IV D) so that their infinitesimal actions are generated, through Poisson brackets, by some observables. We shall use this feature to identify the fundamental observables for appropriately defined elementary systems (material particles will be in the localizable subclass of these systems); their system algebras may then be taken as those generated by the fundamental observables. Employing appropriate tensor products, one then has a *canonical* procedure for setting up the system algebras for systems of a finite number of particles.

A point worth noting is the generality of the reasoning employed above. We did not restrict ourselves to any distinguished class of physical systems (particles, fields,...) nor did we talk about space as the arena for all dynamics. In fact, the formalism evolved will be general enough to permit, in principle, construction of theories in which one starts (for the treatment of the dynamics of the universe as a whole) with matter and its dynamics and space as an arena appears in the description of a later stage in its evolution.

III. SUPERDERIVATION-BASED DIFFERENTIAL CALCULUS; SYMPLECTIC STRUCTURES

In this section we describe a scheme of non-commutative differential calculus which is a superalgebraic version of Dubois-Violette's scheme of noncommutative geometry (henceforth referred to as DVNCG). The induced mappings on (super-)derivations and differential forms (Φ_* and Φ^*) [69,2], which play an important role in the scheme of mechanics to be developed, are treated in some detail. A generalization of DVNCG, which replaces a superalgebra $\mathcal{A}$ by a pair $(\mathcal{A}, \mathcal{X})$ (where $\mathcal{X}$ is a Lie sub-superalgebra of the Lie superalgebra SDer($\mathcal{A}$) of superderivations of $\mathcal{A}$) as the basic entity, is also described. This generalization will be employed in the treatment of general quantum systems admitting superselection rules.

Note. In most applications of supmech, the non-super version of the formalism developed below is adequate; this can be obtained by simply putting,in the formulas below, all the epsilons representing parities equal to zero and all the etas equal to one. Ref [63] contains a brief account of the non-super version.

A. Superalgebras and superderivations

In superalgebra [70-73], all mathematical structures for which addition is defined (vector spaces, algebras, derivations, differential forms etc), have a Z_2-grading where $Z_2 = Z/2Z = \{0, 1\}$. This means that each element of such a structure can be uniquely written as a sum of two parts each of which is assigned a definite parity (0 or 1; correspondingly it is called even or odd). Elements with definite parity are called homogeneous. When a multiplicative operation is defined between homogeneous elements of the same or different mathematical types, the product is a homogeneous element (of appropriate mathematical type) and its parity is the sum (mod 2) of the parities of the multiplicands. We shall denote the parity of a homogeneous object w by $\epsilon(w)$ or ϵ_w according to convenience.

A *supervector space* is a (complex) vector space V admitting a direct sum decomposition $V = V^{(o)} \oplus V^{(1)}$ into spaces of even and odd vectors; a vector $v \in V$ can be uniquely expressed as a sum $v = v_0 + v_1$ of even and odd vectors. A *superalgebra* $\mathcal{A}$ is a supervector space which is an associative algebra with identity; it becomes a **-superalgebra* if an antilinear *-operation or involution $* : \mathcal{A} \to \mathcal{A}$ is defined which satisfies the relations

$$(AB)^* = \eta_{AB} B^* A^*, \quad (A^*)^* = A, \quad I^* = I$$

where I is the identity element and $\eta_{AB} = (-1)^{\epsilon_A \epsilon_B}$. An element $A \in \mathcal{A}$ will be called *hermitian* if $A^* = A$.

The *supercommutator* of two elements A,B of a superalgebra is defined as $[A, B] = AB - \eta_{AB} BA$. For ordinary (anti-)commutators, we shall employ the notations $[A, B]_\mp = AB \mp BA$. A superalgebra $\mathcal{A}$ is said to be *supercommutative* if the supercommutator of every pair of its elements vanishes.

The *graded center* of $\mathcal{A}$, denoted as $Z(\mathcal{A})$, consists of those elements of $\mathcal{A}$ which have vanishing supercommutators with all elements of $\mathcal{A}$; it is clearly a supercommutative superalgebra. Writing $Z(\mathcal{A}) = Z_0(\mathcal{A}) \oplus Z_1(\mathcal{A})$, the object $Z_0(\mathcal{A})$ is the traditional center of $\mathcal{A}$.

A *(*-)homomorphism* of a superalgebra $\mathcal{A}$ into $\mathcal{B}$ is a linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ which preserves products, identity elements, parities (and involutions) :

$$\Phi(AB) = \Phi(A)\Phi(B), \quad \Phi(I_{\mathcal{A}}) = I_{\mathcal{B}}, \quad \epsilon(\Phi(A)) = \epsilon(A), \quad \Phi(A^*) = (\Phi(A))^*;$$

if it is, moreover, bijective, it is called a *(*-)isomorphism.*

A *Lie superalgebra* is a supervector space $\mathcal{L}$ with a *superbracket* operation $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which is (i) bilinear, (ii) graded skew-symmetric which means that, for any two homogeneous elements $a, b \in \mathcal{L}, [a, b] = -\eta_{ab}[b, a]$ and (iii) satisfies the *super Jacobi identity*

$$[a, [b, c]] = [[a, b], c] + \eta_{ab}[b, [a, c]].$$

A (homogeneous) *superderivation* of a superalgebra $\mathcal{A}$ is a linear map $X : \mathcal{A} \to \mathcal{A}$ such that $X(AB) = X(A)B + \eta_{XA} AX(B)$; this is the superalgebraic generalization of the concept of derivation of an algebra. Introducing the multiplication operator μ on $\mathcal{A}$ defined as $\mu(A)B =$

AB, the (necessary and sufficient) condition that a linear map $X : \mathcal{A} \to \mathcal{A}$ is a superderivation may be expressed as

$$X \circ \mu(A) - \eta_{XA}\mu(A) \circ X = \mu(X(A)). \tag{1}$$

Proof. In the equation defining the superderivation X above, express every term as a sequence of mappings acting on the element B; the resulting equation is precisely the equation obtained by operating each side of Eq.(1) on B. □

The set of all superderivations of $\mathcal{A}$ constitutes a Lie superalgebra SDer($\mathcal{A}$) [$= SDer(\mathcal{A})^{(0)} \oplus SDer(\mathcal{A})^{(1)}$]; this is the superalgebraic generalization of the Lie algebra Der($\mathcal{A}$) of all derivations of the algebra $\mathcal{A}$. The *inner superderivations* D_A defined by $D_A B = [A, B]$ are easily seen to satisfy the relation

$$[D_A, D_B] = D_{[A,B]}$$

and constitute a Lie sub-superalgebra ISDer($\mathcal{A}$) of SDer($\mathcal{A}$).

As in DVNCG, it will be implicitly assumed that the superalgebras being employed have a reasonably rich supply of superderivations so that various constructions involving them have a nontrivial content. Some discussion and useful results relating to the precise characterization of the relevant class of algebras may be found in Ref.[74].

The following two facts involving the graded center and the superderivations will be useful [in proving the subcomplex property of $\Omega(\mathcal{A})$ below]:

(i) If $K \in Z(\mathcal{A})$, then $X(K) \in Z(\mathcal{A})$ for all $X \in SDer(\mathcal{A})$.

(ii) For any $K \in Z(\mathcal{A})$ and $X, Y \in SDer(\mathcal{A})$, we have

$$[X, KY] = X(K)Y + \eta_{XK}K[X, Y]. \tag{2}$$

Proof. (i) Expand the two sides of the the relation $X(AK) = \eta_{AK}X(KA)$ (for any $A \in \mathcal{A}$) and cancel the terms containing K on the two sides.
(ii) Expand $[X, KY](A)$ (for any $A \in \mathcal{A}$). □

Corollary. $SDer(\mathcal{A})$ is a $Z(\mathcal{A})$-module.

An involution * on $SDer(\mathcal{A})$ is defined by the relation $X^*(A) = [X(A^*)]^*$. We have

(i) $[X, Y]^* = [X^*, Y^*]$; $(ii)(D_A)^* = -D_{A^*}$.

Proof. In each case, apply the left hand side to a general element $B \in \mathcal{A}$ and follow the definitions. [For an illustration of the kind of steps involved, see the proof of the equations (4) below.] The minus sign in the second relation appears because the definition of D_A involves a supercommutator and a *-operation on a product reverses the order of elements in the product. □

A superalgebra-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ induces a mapping

$$\Phi_* : SDer(\mathcal{A}) \to SDer(\mathcal{B}) \text{ given by } (\Phi_* X)(B) = \Phi(X[\Phi^{-1}(B)]) \tag{3}$$

for all $X \in SDer(\mathcal{A})$ and $B \in \mathcal{B}$. It is the analogue (and a generalization) of the push-forward mapping induced by a diffeomorphism between two manifolds on the vector fields and satisfies the expected relations (with $\Psi : \mathcal{B} \to \mathcal{C}$)

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*; \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y]. \tag{4}$$

Proof : (i) For any $X \in Sder(\mathcal{A})$ and $C \in \mathcal{C}$,

$$\begin{aligned} [(\Psi \circ \Phi)_* X](C) &= (\Psi \circ \Phi)(X[(\Psi \circ \Phi)^{-1}(C)]) \\ &= \Psi[\Phi(X[\Phi^{-1}(\Psi^{-1}(C))])] \\ &= \Psi[(\Phi_* X)(\Psi^{-1}(C)] \\ &= [\Psi_*(\Phi_* X)](C). \end{aligned}$$

(ii) For any $B \in \mathcal{B}$

$$\begin{aligned} (\Phi_*[X, Y])(B) &= \Phi([X, Y](\Phi^{-1}(B))) \\ &= \Phi[X(Y(\Phi^{-1}(B))) - \eta_{XY} Y(X(\Phi^{-1}(B))]. \end{aligned}$$

Now insert $\Phi^{-1} \circ \Phi$ between X and Y in each of the two terms on the right and follow the obvious steps. $\square$.

Note that, Φ_* is a Lie superalgebra isomorphism (i.e. it is bijective and linear and preserves superbrackets).

B. The cochain complex C($SDer(\mathcal{A})$, $\mathcal{A}$)

In DVNCG, one starts with a complex associative algebra $\mathcal{A}$ and constructs a *differential calculus* on it which means a formalism involving differential form like objects on $\mathcal{A}$ with analogues of exterior product, exterior derivative and involution defined on them. For non-commutative $\mathcal{A}$, the choice of differential calculus is not unique; a systematic discussion of the variety of choices may be found in Ref[38]. In applications of NCG, one makes a choice according to convenience. Our needs are best served by a DVNCG type formalism.

For the constructions involving the superalgebraic generalization of DVNCG given in this subsection, some relevant background is provided in Ref.[38,75,76,73]. Grosse and Reiter [75] have given a detailed treatment of the differential geometry of graded matrix algebras. Some related work on supermatrix geometry has also appeared in Ref[77,78]; however, the approach adopted below is closer to Ref[75].

The central object in our scheme is a superalgebra $\mathcal{A}$ [complex, associative, unital (i.e. possessing a unit element), not necessarily supercommutative]; it is the counterpart of $C^\infty(M)$, the commutative algebra of complex smooth functions on the manifold M, in commutative geometry. The Lie superalgebra $SDer(\mathcal{A})$ is the analogue of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M.

Recalling that, in the commutative differential geometry, the differential p-forms are defined as skew-symmetric multilinear maps of $\mathcal{X}(M)^p$ into $C^\infty(M)$, the natural first choice for the

space of (noncommutative) differential p-forms is the space

$$C^p(SDer\mathcal{A},\mathcal{A})[= C^{p,0}(SDer(\mathcal{A}),\mathcal{A}) \oplus C^{p,1}(SDer(\mathcal{A}),\mathcal{A})]$$

of graded skew-symmetric multilinear maps (for $p \geq 1$) of $[SDer(\mathcal{A})]^p$ into $\mathcal{A}$ (the space of $\mathcal{A}$-valued p-cochains of $SDer(\mathcal{A})$; it is the super-analogue of the Chevalley-Eilenberg p-cochain space [79]). We have $C^0(SDer(\mathcal{A}),\mathcal{A}) = \mathcal{A}$. For $\omega \in C^{p,s}(SDer(\mathcal{A}),\mathcal{A})$, we have

$$\omega(..,X,Y,..) = -\eta_{XY}\omega(..,Y,X,..). \tag{5}$$

For a general permutation σ of the arguments of ω, we have

$$\omega(X_{\sigma(1)},..,X_{\sigma(p)}) = \kappa_\sigma \gamma_p(\sigma;\epsilon_{X_1},..,\epsilon_{X_p})\omega(X_1,..,X_p) \tag{6}$$

where κ_σ is the parity of the permutation σ and

$$\gamma_p(\sigma;s_1,..,s_p) = \prod_{\substack{j,k=1,..,p;\\ j<k,\sigma^{-1}(j)>\sigma^{-1}(k)}} (-1)^{s_j s_k}. \tag{7}$$

An involution * on the cochains is defined by the relation $\omega^*(X_1,..,X_p) = [\omega(X_1^*,..,X_p^*)]^*$; ω is said to be real (imaginary) if $\omega^* = \omega(-\omega)$.

The *exterior product*

$$\wedge : C^{p,r}(SDer(\mathcal{A}),\mathcal{A}) \times C^{q,s}(SDer(\mathcal{A}),\mathcal{A}) \to C^{p+q,r+s}(SDer(\mathcal{A}),\mathcal{A})$$

is defined as

$$(\alpha\wedge\beta)(X_1,..,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}} \kappa_\sigma\gamma_{p+q}(\sigma;\epsilon_{X_1},..,\epsilon_{X_{p+q}})(-1)^{s\sum_{j=1}^p \epsilon_{X_{\sigma(j)}}} \alpha(X_{\sigma(1)},..,X_{\sigma(p)})\beta(X_{\sigma(p+1)},..,X_{\sigma(p+q)}). \tag{8}$$

With this product, the graded vector space

$$C(SDer(\mathcal{A}),\mathcal{A}) = \bigoplus_{p\geq 0} C^p(SDer(\mathcal{A}),\mathcal{A})$$

becomes an $N_0 \times Z_2$-bigraded complex algebra. (Here N_0 is the set of non-negative integers.)

The Lie superalgebra $SDer(\mathcal{A})$ acts on itself and on C($SDer(\mathcal{A})$,$\mathcal{A}$) through *Lie derivatives*. For each $Y \in SDer(\mathcal{A})^{(r)}$, one defines linear mappings $L_Y : SDer(\mathcal{A})^{(s)} \to SDer(\mathcal{A})^{(r+s)}$ and $L_Y : C^{p,s}(SDer(\mathcal{A}),\mathcal{A}) \to C^{p,r+s}(SDer(\mathcal{A}),\mathcal{A})$ such that the following three conditions hold :

$$L_Y(A) = Y(A) \quad \text{for all } A \in \mathcal{A} \tag{9}$$

$$L_Y[X(A)] = (L_Y X)(A) + \eta_{XY} X[L_Y(A)] \tag{10}$$

$$L_Y[\omega(X_1,..,X_p)] = (L_Y\omega)(X_1,..,X_p) + \sum_{i=1}^{p}(-1)^{\epsilon_Y(\epsilon_\omega+\epsilon_{X_1}+..+\epsilon_{X_{i-1}})}.$$
$$.\omega(X_1,..,X_{i-1},L_YX_i,X_{i+1},..,X_p). \tag{11}$$

The first two conditions give

$$L_YX = [Y,X] \tag{12}$$

which, along with the third, gives

$$(L_Y\omega)(X_1,..,X_p) = Y[\omega(X_1,..,X_p)] - \sum_{i=1}^{p}(-1)^{\epsilon_Y(\epsilon_\omega+\epsilon_{X_1}+..+\epsilon_{X_{i-1}})}.$$
$$.\omega(X_1,..,X_{i-1},[Y,X_i],X_{i+1},..,X_p). \tag{13}$$

Two important properties of the Lie derivative are, in obvious notation,

$$[L_X,L_Y] = L_{[X,Y]} \tag{14}$$

$$L_Y(\alpha\wedge\beta) = (L_Y\alpha)\wedge\beta + \eta_{Y\alpha}\ \alpha\wedge(L_Y\beta). \tag{15}$$

For each $X \in SDer(\mathcal{A})^{(r)}$, we define the *interior product*
$i_X : C^{p,s}(SDer(\mathcal{A}),\mathcal{A}) \to C^{p-1,r+s}(SDer(\mathcal{A}),\mathcal{A})$ (for $p \geq 1$) by

$$(i_X\omega)(X_1,..,X_{p-1}) = \omega(X,X_1,..,X_{p-1}). \tag{16}$$

One defines $i_X(A) = 0$ for all $A \in \mathcal{A}$. Note that there is no $\eta_{X\omega}$ factor on the right in Eq.(16). A more appropriate notation (from the point of view of proper/unambiguous placing of symbols) for $i_X\omega$ is ω_X. (See Ref[80].) Some important properties of the interior product are :

$$i_X\circ i_Y + \eta_{XY}i_Y\circ i_X = 0 \tag{17}$$

$$i_X(\alpha\wedge\beta) = \eta_{X\beta}(i_X\alpha)\wedge\beta + (-1)^p\alpha\wedge(i_X\beta) \tag{18}$$

$$(L_Y\circ i_X - i_X\circ L_Y) = \eta_{X\omega}i_{[X,Y]}\omega. \tag{19}$$

The *exterior derivative* $d : C^{p,r}(SDer(\mathcal{A}),\mathcal{A}) \to C^{p+1,r}(SDer(\mathcal{A}),\mathcal{A})$ is defined through the relation

$$(i_X\circ d + d\circ i_X)\omega = \eta_{X\omega}\ L_X\omega. \tag{20}$$

For p = 0, it takes the form

$$(dA)(X) = \eta_{XA}\ X(A). \tag{21}$$

Taking, in Eq.(20), ω a homogeneous p-form and contracting both sides with homogeneous derivations $X_1, .., X_p$ gives the quantity $(d\omega)(X, X_1, .., X_p)$ in terms of evaluations of exterior derivatives of lower degree forms. This determines $d\omega$ recursively giving

$$(d\omega)(X_0, X_1, .., X_p) = \sum_{i=0}^{p} (-1)^{i+a_i} X_i[\omega(X_0, .., \hat{X}_i, .., X_p)]$$
$$+ \sum_{0 \le i < j \le p} (-1)^{j+b_{ij}} \omega(X_0, .., X_{i-1}, [X_i, X_j], X_{i+1}, .., \hat{X}_j, .., X_p) \tag{22}$$

where the hat indicates omission and

$$a_i = \epsilon_{X_i}(\epsilon_\omega + \sum_{k=0}^{i-1} \epsilon_{X_k});\ b_{ij} = \epsilon_{X_j} \sum_{k=i+1}^{j-1} \epsilon_{X_k}.$$

Some important properties of the exterior derivative are

$$d^2 (= d \circ d) = 0 \tag{23}$$

$$d \circ L_Y = L_Y \circ d \tag{24}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) \tag{25}$$

where α is a p-cochain. Eq.(23) shows that the pair, (C($SDer(\mathcal{A})$, $\mathcal{A}$), d) constitutes a cochain complex. We shall call a cochain α *closed* if $d\alpha = 0$ and *exact* if $\alpha = d\beta$ for some cochain β.

C. Differential forms

Taking clue from Ref[38] [where the subcomplex of $Z(\mathcal{A})$-linear cochains (Z($\mathcal{A}$) being, in the notation of Ref[38], the center of the algebra $\mathcal{A}$) was adopted as the space of differential forms], we consider the subset $\Omega(\mathcal{A})$ of $C(SDer(\mathcal{A}), \mathcal{A})$ consisting of $Z_0(\mathcal{A})$-linear cochains. Eq.(2) ensures that this subset is closed under the action of d and, therefore, a subcomplex. We shall take this space to be the space of differential forms in subsequent geometrical work. We have, of course,

$$\Omega(\mathcal{A}) = \oplus_{p \ge 0} \Omega^p(\mathcal{A})$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^p(\mathcal{A}) = \Omega^{p,0}(\mathcal{A}) \oplus \Omega^{p,1}(\mathcal{A})$ for all $p \ge 0$.

D. Induced mappings on differential forms

A superalgebra *-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ induces, besides the Lie superalgebra-isomorphism $\Phi_* : SDer(\mathcal{A}) \to SDer(\mathcal{B})$, a mapping

$$\Phi^* : C^{p,s}(SDer\mathcal{B}, \mathcal{B}) \to C^{p,s}(SDer(\mathcal{A}), \mathcal{A})$$

given, in obvious notation, by

$$(\Phi^* \omega)(X_1, .., X_p) = \Phi^{-1}[\omega(\Phi_* X_1, .., \Phi_* X_p)]. \tag{26}$$

The maping Φ preserves (bijectively) all the algebraic relations. Eq.(3) shows that Φ_* preserves $Z_0(\mathcal{A})$-linear combinations of the superderivations. It follows that Φ^* maps differntial forms onto differential forms.

These mappings are analogues (and generalizations) of the pull-back mappings on differential forms (on manifolds) induced by diffeomorphisms. They satisfy the expected relations [with $\Psi : \mathcal{B} \to \mathcal{C}$]

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \tag{27}$$

$$\Phi^*(\alpha \wedge \beta) = (\Phi^*\alpha) \wedge (\Phi^*\beta) \tag{28}$$

$$\Phi^*(d\alpha) = d(\Phi^*\alpha). \tag{29}$$

Outlines of proofs of Eqs.(27-29) :

Eq.(27) : For $\omega \in C^{p,s}(Sder(\mathcal{C}), \mathcal{C})$ and $X_1, .., X_p \in Sder(\mathcal{A})$,

$$\begin{aligned}
[(\Psi \circ \Phi)^*\omega](X_1, .., X_p) &= (\Phi^{-1} \circ \Psi^{-1})[\omega(\Psi_*(\Phi_* X_1), .., \Psi_*(\Phi_* X_p))] \\
&= \Phi^{-1}[(\Psi^*\omega)(\Phi_* X_1, .., \Phi_* X_p)] \\
&= \Phi^*(\Psi^*\omega)(X_1, .., X_p). \ \square
\end{aligned}$$

Eq.(28) : For $\alpha \in C^{p,r}(SDer(\mathcal{B}), \mathcal{B}), \beta \in C^{q,s}(Sder(\mathcal{B}), \mathcal{B})$ and $X_1, .., X_{p+q} \in SDer(\mathcal{A})$,

$$[\Phi^*(\alpha \wedge \beta)](X_1, .., X_{p+q}) = \Phi^{-1}[(\alpha \wedge \beta)(\Phi_* X_1, .., \Phi_* X_{p+q})].$$

Expanding the wedge product and noting that

$$\Phi^{-1}[\alpha(..)\beta(..)] = \Phi^{-1}[\alpha(..)].\Phi^{-1}[\beta(..)],$$

the right hand side is easily seen to be equal to $[(\Phi^*\alpha) \wedge (\Phi^*\beta)](X_1, .., X_{p+q})$. $\square$

Eq.(29) : We have

$$[\Phi^*(d\alpha)](X_0, .., X_p) = \Phi^{-1}[(d\alpha)(\Phi_* X_0, .., \Phi_* X_p)].$$

Using Eq.(22) for $d\alpha$ and noting that

$$\begin{aligned}
\Phi^{-1}[(\Phi_* X_i)(\alpha(\Phi_* X_0, ..)) &= \Phi^{-1}[\Phi(X_i[\Phi^{-1}(\alpha(\Phi_* X_0, ..))]] \\
&= X_i[(\Phi^*\alpha)(X_0, ..)]
\end{aligned}$$

and making similar (in fact, simpler) manipulations with the second term in the expression for $d\alpha$, it is easily seen that the left hand side of Eq.(29), evaluated at $(X_0, .., X_p)$, equals $[(d(\Phi^*\alpha)](X_0, .., X_p)$. $\square$

Now, let $\Phi_t : \mathcal{A} \to \mathcal{A}$ be a one-parameter family of transformations (i.e. superalgebra isomorphisms) given, for small t, by

$$\Phi_t(A) \simeq A + tg(A) \tag{30}$$

where g is some (linear, even) mapping of $\mathcal{A}$ into itself. The condition $\Phi_t(AB) = \Phi_t(A)\Phi_t(B)$ gives $g(AB) = g(A)B + Ag(B)$ implying that $g(A) = Y(A)$ for some even superderivation Y of $\mathcal{A}$ (to be called the *infinitesimal generator* of Φ_t). From Eq.(3), we have, for small t,

$$(\Phi_t)_* X \simeq X + t[Y, X] = X + tL_Y X. \tag{31}$$

Similarly, for any p-form ω, we have

$$\Phi_t^* \omega \simeq \omega - tL_Y \omega. \tag{32}$$

Proof : We have

$$\begin{aligned}
(\Phi_t^*\omega)(X_1, .., X_p) &= \Phi_t^{-1}[\omega\,((\Phi_t)_* X_1, ..(\Phi_t)_* X_p)] \\
&\simeq \omega(X_1, .., X_p) - tY\omega(X_1, .., X_p) \\
&\quad + t\sum_{i=1}^{p} \omega(X_1, .., [Y, X_i], .., X_p) \\
&= [\omega - tL_Y\omega]](X_1, .., X_p). \quad \square
\end{aligned}$$

Note that (as in the commutative geometry), the Lie derivative term appears with a plus sign in Eq.(31) and a minus sign in Eq.(32). This is because, in the pull-back action, the effective mapping is Φ_t^{-1}.

E. A generalization of the DVNCG scheme [69,2]

In the formula (22) for $d\omega$, the superderivations X_j appear on the right either singly or as supercommutators. It should, therefore, be possible to restrict them to a Lie sub-superalgebra $\mathcal{X}$ of $SDer(\mathcal{A})$ and develop the whole formalism with the pair $(\mathcal{A}, \mathcal{X})$ obtaining thereby a useful generalization of the formalism developed in the previous three subsections. Working with such a pair is the analogue of restricting oneself to a leaf of a foliated manifold as the first example below indicates.

Examples : (i) $\mathcal{A} = C^\infty(R^3)$; $\mathcal{X}$= the Lie subalgebra of the Lie algebra $\mathcal{X}(R^3)$ of vector fields on R^3 generated by the Lie differential operators $L_j = \epsilon_{jkl} x_k \partial_l$ for the SO(3)-action on R^3. These differential operators, when expressed in terms of the polar coordinates r, θ, ϕ,contain only the partial derivatives with respect to θ and ϕ; they, therefore, act on the 2-dimensional spheres that constitute the leaves of the foliation $R^3 - \{(0,0,0)\} \simeq S^2 \times R$. The restriction [of the pair $(\mathcal{A}, \mathcal{X}(R^3))$] to ($\mathcal{A}$,$\mathcal{X}$) amounts to restricting oneself to a leaf (S^2) in the present case.

(ii) $\mathcal{A} = M_4(C)$, the algebra of complex 4×4 matrices. The vector space C^4 on which these matrices act serves as the carrier space of the spin $s = \frac{3}{2}$ projective irreducible representation

of the rotation group SO(3). Denoting by S_j (j=1,2,3) the 4×4 matrices representing the generators of the Lie algebra so(3), let $\mathcal{X}$ be the real Lie algebra generated by the inner derivations D_{S_j}. The pair ($\mathcal{A}$, $\mathcal{X}$) is relevant for the treatment of spin dynamics for $s = \frac{3}{2}$.

A formalism analogous to that of sections III B and III C can be developed by working with a pair ($\mathcal{A}$,$\mathcal{X}$) as mentioned above. One obtains the cochains $C^{p,s}(\mathcal{X}, \mathcal{A})$ for which the formulas of these sections are valid (with the $X_j s$ restricted to $\mathcal{X}$). The differential forms $\Omega^{p,s}(\mathcal{A})$ will now be replaced by the objects $\Omega^{p,s}(\mathcal{X}, \mathcal{A})$ obtained by restricting the cochains to the $Z_0(\mathcal{A})$-linear ones. [In the new notation, the objects $\Omega^{p,s}(\mathcal{A})$ will be called $\Omega^{p,s}(SDer(\mathcal{A}), \mathcal{A})$.]

Note. In Ref[2], the notation $\Omega^p(\mathcal{A}, \mathcal{X})$ was used for the space of differential p-forms [which appeared natural in view of the notation ($\mathcal{A}$, $\mathcal{X}$) for the pairs called algebraic differential systems there]. In the present work, we have changed it to $\Omega^p(\mathcal{X}, \mathcal{A})$ to bring it in tune with the notation for Lie algebra cochains in the mathematics literature.

To define the induced mappings Φ_* and Φ^* in the present context, one should employ a *pair-isomorphism* $\Phi : (\mathcal{A}, \mathcal{X}) \to (\mathcal{B}, \mathcal{Y})$ which consists of a superalgebra *- isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that the induced mapping $\Phi_* : SDer(\mathcal{A}) \to SDer(\mathcal{B})$ restricts to an isomorphism of $\mathcal{X}$ onto $\mathcal{Y}$. Various properties of the induced mappings hold as before.

Given a one-parameter family of transformations $\Phi_t : (\mathcal{A}, \mathcal{X}) \to (\mathcal{A}, \mathcal{X})$, the condition $(\Phi_t)_* \mathcal{X} \subset \mathcal{X}$ implies that the infinitesimal generator Y of Φ_t must satisfy the condition $[Y, X] \in \mathcal{X}$ for all $X \in \mathcal{X}$. In practical applications one will generally have $Y \in \mathcal{X}$ which trivially satisfies this condition.

This generalization will be used in sections IV E and VII E below.

F. Symplectic structures

Note. The sign conventions about various quantities adopted below are parallel to those of Woodhouse [81]. This results in a (super-) Poisson bracket which (when applied to classical Hamiltonian mechanics) gives one differing from the Poisson bracket in most current books on mechanics by a minus sign. [See Eq.(46).] The main virtue of the adopted conventions is that Eq.(38) below has no unpleasant minus sign.

A *symplectic structure* on a superalgebra $\mathcal{A}$ is a 2- form ω (the *symplectic form*) which is even, closed and *non-degenerate* in the sense that, for every $A \in \mathcal{A}$, there exists a unique superderivation Y_A in $SDer(\mathcal{A})$ [the *(globally) Hamiltonian superderivation* corresponding to A] such that

$$i_{Y_A} \omega = -dA. \tag{33}$$

The pair $(\mathcal{A}, \omega)$ will be called a *symplectic superalgebra*. A symplectic structure is said to be *exact* if the symplectic form is exact ($\omega = d\theta$ for some 1-form θ called the *symplectic potential*).

A *symplectic mapping* from a symplectic superalgebra $(\mathcal{A}, \alpha)$ to another one $(\mathcal{B}, \beta)$ is a superalgebra isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\Phi^* \beta = \alpha$. (If the symplectic structures involved

are exact, one requires a symplectic mapping to preserve the symplectic potential under the pull-back action; the commutation of the exterior derivative with the pull-back action then guarantees the preservation of the symplectic form.) A symplectic mapping from a symplectic superalgebra onto itself will be called a *canonical/symplectic transformation*. The symplectic form and its exterior powers are invariant under canonical transformations.

If Φ_t is a one-parameter family of canonical transformations generated by $X \in SDer(\mathcal{A})$, the condition $\Phi_t^* \omega = \omega$ implies, with Eq.(32),

$$L_X \omega = 0. \tag{34}$$

The argument just presented gives Eq.(34) with X an even superderivation. More generally, a superderivation X (even or odd or inhomogeneous) satisfying Eq.(34) will be called a *locally Hamiltonian* superderivation. Eq.(20) and the condition $d\omega = 0$ imply that Eq.(34) is equivalent to the condition

$$d(i_X \omega) = 0. \tag{35}$$

The (globally) Hamiltonian superderivations defined by Eq(33) constitute a subclass of locally Hamiltonian superderivations for which $i_X \omega$ is exact. Note from Eq(33) that $\epsilon(Y_A) = \epsilon(A)$. In analogy with the commutative case, the supercommutator of two locally Hamiltonian superderivations is a globally Hamiltonian superderivation. Indeed, given two locally Hamiltonian superderivations X and Y, we have, recalling Equations (19) and (20),

$$\begin{aligned} \eta_{X\omega} i_{[X,Y]} \omega &= (L_Y \circ i_X - i_X \circ L_Y)\omega \\ &= \eta_{Y\omega}(i_Y \circ d + d \circ i_Y)(i_X \omega) \\ &= \eta_{Y\omega} d(i_Y i_X \omega) \end{aligned}$$

which is exact. It follows that the locally Hamiltonian superderivations constitute a Lie superalgebra in which the globally Hamiltonian superderivations constitute an ideal.

The *Poisson bracket* (PB) of two elements A and B of $\mathcal{A}$ is defined as

$$\{A, B\} = \omega(Y_A, Y_B) = Y_A(B) = -\eta_{AB} Y_B(A). \tag{36}$$

It obeys the superanalogue of the Leibnitz rule :

$$\begin{aligned} \{A, BC\} = Y_A(BC) &= Y_A(B)C + \eta_{AB} B Y_A(C) \\ &= \{A, B\}C + \eta_{AB} B\{A, C\}. \end{aligned} \tag{37}$$

As in the classical case, we have the relation

$$[Y_A, Y_B] = Y_{\{A,B\}}. \tag{38}$$

Eqn.(38) follows by using the equation for $i_{[X,Y]}\omega$ above with $X = Y_A$ and $Y = Y_B$ and Eq.(33), remembering that Eq.(33) determines Y_A uniquely. The super-Jacobi identity

$$\begin{aligned} 0 &= \frac{1}{2}(d\omega)(Y_A, Y_B, Y_C) \\ &= \{A, \{B, C\}\} + (-1)^{\epsilon_A(\epsilon_B+\epsilon_C)}\{B, \{C, A\}\} \\ &\quad +(-1)^{\epsilon_C(\epsilon_A+\epsilon_B)}\{C, \{A, B\}\} \end{aligned} \tag{39}$$

is obtained by using Eqn.(22) and noting that

$$\begin{aligned} Y_A[\omega(Y_B, Y_C)] &= \{A, \{B, C\}\} \\ \omega([Y_A, Y_B], Y_C) &= \omega(Y_{\{A,B\}}, Y_C) = \{\{A, B\}, C\}. \end{aligned}$$

Clearly, the pair ($\mathcal{A}$, $\{,\ \}$) is a Lie superalgebra. Eq.(38) shows that the linear mapping $A \mapsto Y_A$ is a Lie-superalgebra homomorphism.

An element A of $\mathcal{A}$ can act, via Y_A, as the infinitesimal generator of a one-parameter family of canonical transformations. The change in $B \in \mathcal{A}$ due to such an infinitesimal transformation is

$$\delta B = \epsilon Y_A(B) = \epsilon\{A, B\}. \tag{40}$$

In particular, if $\delta B = \epsilon I$ (infinitesimal 'translation' in B), we have

$$\{A, B\} = I \tag{41}$$

which is the noncommutative analogue of the classical PB relation $\{p, q\} = 1$. A pair (A,B) of elements of $\mathcal{A}$ satisfying the condition (41) will be called a *canonical pair*.

G. Reality properties of the symplectic form and the Poisson bracket

For classical superdynamical systems, conventions about reality properties of the symplectic form are based on the fact [82,83] that the Lagrangian is a real, even object. The matrix of the symplectic form is then real- antisymmetric in the 'bosonic sector' and imaginary-symmetric in the 'fermionic sector' (which means anti-Hermitian in both sectors). Keeping this in view, we impose, in supmech, the following reality condition on the symplectic form ω:

$$\omega^*(X, Y) = -\eta_{XY}\omega(Y, X) \quad \text{for all } X, Y \in SDer(\mathcal{A}); \tag{42}$$

but this means,by Eq.(5), that $\omega^* = \omega$ (i.e. ω is real) which is the most natural assumption to make about ω. Eq.(42) is equivalent to the condition

$$\omega(X^*, Y^*) = -\eta_{XY}[\omega(Y, X)]^*. \tag{43}$$

Now, for arbitrary $A, B \in \mathcal{A}$, we have

$$\begin{aligned} \{A, B\}^* &= Y_A(B)^* = Y_A^*(B^*) = \eta_{AB}dB^*(Y_A^*) \\ &= -\eta_{AB}\omega(Y_{B^*}, Y_A^*) = [\omega(Y_A, Y_{B^*}^*]^* \\ &= -[dA(Y_{B^*}^*)]^* = -\eta_{AB}[Y_{B^*}^*(A)]^* \\ &= -\eta_{AB}Y_{B^*}(A^*) \end{aligned}$$

giving finally

$$\{A, B\}^* = -\eta_{AB}\{B^*, A^*\}. \tag{44}$$

Eq.(44) is consistent with the reality properties of the classical and quantum Poisson brackets.[See equations (46) and (54) below.]

H. The algebra $\mathcal{A}_{cl}$ and the classical Poisson bracket

Classical symplectic structure, traditionally treated in the framework of a symplectic manifold (M, ω_{cl}) [where M is a differentiable manifold of even dimension (say, 2n) and ω_{cl} , the classical symplectic form, a nondegenerate differential 2-form on M], can be realized as a special case of the symplectic structure treated above by taking $\mathcal{A} = C^\infty(M, C) \equiv \mathcal{A}_{cl}$, the commutative algebra of smooth complex-valued functions on M [with pointwise product (fg)(x) = f(x)g(x)]. The star operation in this case is the complex conjugation : $f^*(x) = \overline{f(x)}$. The derivations of $\mathcal{A}_{cl}$ are the smooth complex vector fields. The differential forms of subsection C above are easily seen, for $\mathcal{A} = \mathcal{A}_{cl}$, to be the traditional differential forms on M. The symplectic structure is defined in terms of the classical differential form (in canonical coordinates)

$$\omega_{cl} = \sum_{j=1}^{n} dp_j \wedge dq_j. \tag{45}$$

In terms of the general local coordinates $\xi^a (a = 1, .., 2n)$, writing $\omega_{cl} = (\omega_{cl})_{ab} d\xi^a \wedge d\xi_b$, the classical PB is given by

$$\{f, g\}_{cl} = \omega_{cl}^{ab} \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b}$$

where (ω_{cl}^{ab}) is the inverse of the matrix $((\omega_{cl})_{ab})$; in canonical coordinates, it takes the traditional form

$$\{f, g\}_{cl} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right). \tag{46}$$

Noting that the form ω_{cl} and the coordinates ξ^a are real, the reality properties embodied in the equations (42) and (44) are obvious in the present case.

I. Special algebras; the canonical symplectic form

In this subsection, we shall consider a distinguished class of superalgebras [69,38,2] which have a canonical symplectic structure associated with them. As we shall see in section VII, these superalgebras play an important role in Quantum mechanics.

A complex, associative, non-supercommutative *-superalgebra will be called *special* if all its superderivations are inner. The differential 2-form ω_c defined on such a superalgebra $\mathcal{A}$ by

$$\omega_c(D_A, D_B) = [A, B] \tag{47}$$

is said to be the *canonical form* on $\mathcal{A}$. It is easily seen to be closed [the equation $(d\omega_c)(D_A, D_B, D_C) = 0$ is nothing but the Jacobi identity for the supercommutator], imaginary (i.e. $\omega_c^* = -\omega_c$) and dimensionless. For any $A \in \mathcal{A}$, the equation

$$\omega_c(Y_A, D_B) = -(dA)(D_B) = [A, B]$$

has the unique solution $Y_A = D_A$. (To see this, note that, since all derivations are inner, $Y_A = D_C$ for some $C \in \mathcal{A}$; the condition $\omega_c(D_C, D_B) = [C, B] = [A, B]$ for all $B \in \mathcal{A}$ implies that $C - A \in Z(\mathcal{A})$. But then $D_C = D_A$. QED) This gives

$$i_{D_A}\omega_c = -dA. \tag{48}$$

The form ω_c is, therefore, non-degenerate and defines, on $\mathcal{A}$, the *canonical symplectic structure*. It gives, as Poisson bracket, the supercommutator :

$$\{A, B\} = Y_A(B) = D_A(B) = [A, B]. \tag{49}$$

Using equations (48) and (20), it is easily seen that the form ω_c is *invariant* in the sense that $L_X\omega_c = 0$ for all $X \in SDer(\mathcal{A})$. The invariant symplectic structure on the algebra $M_n(C)$ of complex $n \times n$ matrices obtained by Dubois-Violette and coworkers [37] is a special case of the canonical symplectic structure on special algebras described above.

If, on a special superalgebra $\mathcal{A}$, instead of ω_c, we take $\omega = b\omega_c$ as the symplectic form (where b is a nonzero complex number), we have

$$Y_A = b^{-1}D_A, \qquad \{A, B\} = b^{-1}[A, B]. \tag{50}$$

We shall make use of such a symplectic structure (with $b = -i\hbar$) in the following subsection and in the treatment of quantum mechanics in section VII. (Note that b must be imaginary to make ω real.) Such a symplectic structure with general nonzero b will be referred to as the *quantum symplectic structure with parameter b*.

J. The quantum symplectic form

Let us consider the traditional QM of a non-relativistic spinless particle. The central object in it is the Hilbert space $\mathcal{H} = L^2(R^3, dx)$ of complex square-integrable functions on R^3. The fundamental observables of such a particle are the Cartesian components $X_j, P_j (j = 1, 2, 3)$ of position and momentum vectors which are self-adjoint linear operators represented, in the oft-used Schrödinger representation, as

$$(X_j\phi)(x) = x_j\phi(x); \quad (P_j\phi)(x) = -i\hbar\frac{\partial\phi}{\partial x_j}. \tag{51}$$

These operators satisfy the canonical commutation relations (CCR)

$$[X_j, X_k] = 0 = [P_j, P_k]; \quad [X_j, P_k] = i\hbar I \quad (j, k = 1, 2, 3) \tag{52}$$

where I is the unit operator. The functions ϕ in Eq.(51) must be restricted to a suitable dense domain $\mathcal{D}$ in $\mathcal{H}$ which is generally taken to be the space $\mathcal{S}(R^3)$ of Shwartz functions. Other operators appearing in QM of the particle belong to the algebra $\mathcal{A}$ generated by the operators X_j, P_j (j= 1,2,3) and I [subject to the CCR (52)]. The space $\mathcal{D} = \mathcal{S}(R^3)$ is clearly an invariant domain for all elements of $\mathcal{A}$. Defining a *-operation on $\mathcal{A}$ by $A^* = A^\dagger|\mathcal{D}$, the Hermitian elements of $\mathcal{A}$ represent the general observables of the particle.

The algebra $\mathcal{A}$ obtained above is 'special' [38](in the sense defined in the previous subsection); one has, therefore, a canonical form ω_c defined on it. The *quantum symplectic structure* is defined on $\mathcal{A}$ by employing the *quantum symplectic form*

$$\omega_Q = -i\hbar\omega_c. \tag{53}$$

Note that the factor i serves to make ω real and $\hbar$ to give it the dimension of action (which is the correct dimension of a symplectic form in mechanics). The minus sign is a matter of convention. Eq.(50) now gives the *quantum Poisson bracket*

$$\{A, B\}_Q = (-i\hbar)^{-1}[A, B]. \tag{54}$$

The CCR (52) can now be expressed as the quantum Poisson brackets involving the canonical pairs (X_j, P_j) :

$$\{X_j, X_k\}_Q = 0 = \{P_j, P_k\}_Q; \quad \{P_j, X_k\}_Q = \delta_{jk}I \quad (j, k = 1, 2, 3). \tag{55}$$

IV. THE FORMALISM OF SUPMECH

A. The system algebra and states

Supmech associates, with every physical system, a symplectic superalgebra $(\mathcal{A}, \omega)$ of the type considered in the previous section. Here we shall treat the term 'physical system' informally as is traditionally done; some formalities in this connection will be taken care of in section IX where the axioms are stated. The even Hermitian elements of $\mathcal{A}$ represent *observables* of the system. The collection of all observables in $\mathcal{A}$ will be denoted as $\mathcal{O}(\mathcal{A})$.

To take care of limit processes and continuity of mappings, we must employ topological algebras. The choice of the admissible class of topological algebras must meet the following reasonable requirements:

(i) It should be closed under the formation of (a) topological completions and (b) tensor products. (Both are nontrivial requirements. [51])
(ii) It should include
(a) the Op*-algebras [12] based on the pairs $(\mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{D}$ a dense linear subset of $\mathcal{H}$. [Recall that such an algebra is the algebra of operators which, along with their adjoints, map $\mathcal{D}$ into itself. These are the algebras of operators (not necessarily bounded) appearing in the traditional Hilbert space QM; for example, the algebra $\mathcal{A}$ in section III J belongs to this class.];

(b) noncommutative *-algebras having Gel'fand-Naimark-Segal (GNS) representations in *separable* Hilbert spaces (this is to ensure that the Hilbert spaces corresponding to quantum systems defined initially algebraically are separable);
(c) Algebras of smooth functions on manifolds (to accommodate classical dynamics).

The right choice appears to be the $\hat{\otimes}$-(star-)algebras of Helemskii [84] (i.e. locally convex *-algebras which are complete and Hausdorff with a jointly continuous product) satisfying the additional condition of being separable. [Note. The condition of separability may have to be dropped in applications to quantum field theory.] Henceforth all (super-)algebras employed will be assumed to belong to this class. For easy reference, unital *-algebras of this class will be called *supmech-admissible*. Mappings between topological spaces should henceforth be understood as continuous.

A *state* on a (unital) *-algebra $\mathcal{A}$ (which may or may not be a superalgebra) is a linear functional ϕ on $\mathcal{A}$ which is (i) positive [which means $\phi(A^*A) \geq 0$ for all $A \in \mathcal{A}$) and (ii) normalized [i.e. $\phi(I) = 1$]. Given a state ϕ, the quantity $\phi(A)$ for any observable A is real (this can be seen by considering, for example, the quantity$\phi[(I+A)^*(I+A)]$) and is to be interpreted as the expectation value of A in the state ϕ. Following general usage in literature, we shall call observables of the form A^*A or sum of such terms *positive* (strictly speaking, the term 'non-negative' would be more appropriate); states assign non-negative expectation values to such observables. The family of all states on $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$. It is easily seen to be closed under convex combinations: given $\phi_i \in \mathcal{S}(\mathcal{A})$, i = 1,..,n and $p_i \geq 0$ with $p_1 + .. + p_n = 1$, we have $\phi = \sum_{i=1}^{n} p_i\phi_i$ also in $\mathcal{S}(\mathcal{A})$.[More generally, the integral of an $\mathcal{S}(\mathcal{A})$-valued function on a probability space integrated over the probability measure is an element of $\mathcal{S}(\mathcal{A})$.] States which cannot be expressed as nontrivial convex combinations of other states will be called *pure* states and others *mixed* states or *mixtures*. The family of pure states of $\mathcal{A}$ will be denoted as $\mathcal{S}_1(\mathcal{A})$. The triple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega)$ will be referred to as a *symplectic triple*.

Note. In physics literature, it is sometimes found convenient to include, among states, those for which the magnitudes of expectation values of some observables are infinite. For example, in practical work in QM, one employs wave functions which give infinite values for the expectation values of some unbounded observables like position, momentum or energy. It needs to be made clear, however, that *physical states* (or *physically realizable states*) must be restricted to those for which the expectation values of *all* observables are finite. In practical quantum mechanical work this would mean that only wave functions lying in a common invariant dense domain of all observables must be treated as representing physical states. Our formal definition of state, in fact, allows only physical states. (This is because the domain of definition of the functionals defining states is the whole algebra $\mathcal{A}$.)

In a sensible physical theory, the collection of pure states must be rich enough to distinguish between two different observables. (Mixtures represent averaging over ignorances over and above those implied by the irreducible probabilistic aspect of the theory; they, therefore, are

not the proper objects for a statement of the above sort.) Similarly, there should be enough observables to distinguish between different pure states. These requirements are taken care of in supmech by stipulating that the pair $(\mathcal{O}(\mathcal{A}),\ \mathcal{S}_1(\mathcal{A}))$ be *compatibly complete* in the sense that

(i) given $A, B \in \mathcal{O}(\mathcal{A}), A \neq B$, there should be a state $\phi \in \mathcal{S}_1(\mathcal{A})$ such that $\phi(A) \neq \phi(B)$;

(ii) given two different states ϕ_1 and ϕ_2 in $\mathcal{S}_1(\mathcal{A})$, there should be an $A \in \mathcal{O}(\mathcal{A})$ such that $\phi_1(A) \neq \phi_2(A)$.

We shall refer to this condition as the 'CC condition' for the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$.

Expectation values of all even elements of $\mathcal{A}$ can be expressed in terms of those of the observables (by considering the breakup of such an element into its Hermitian and anti-Hermitian part). This leaves out the odd elements of $\mathcal{A}$. It appears reasonable to demand that, if observables are to be restricted to *even* Hermitian elements of $\mathcal{A}$, the expectation values $\phi(A)$ of all odd elements $A \in \mathcal{A}$ must vanish for all pure states (and, therefore, for all states).

Next, we consider the relation between states and traditional probability measures. We shall introduce classical probabilities in the formalism through a straightforward formalization of a measurement situation. To this end, we introduce a measurable space $(\Omega, \mathcal{F})$ and associate, with every measurable set $E \in \mathcal{F}$, a positive observable $\nu(E)$ such that

$$(i)\ \nu(\emptyset) = 0,\ (ii)\ \nu(\Omega) = I,\ (iii)\ \nu(\cup_i E_i) = \sum_i \nu(E_i)\ \text{(for disjoint unions)}.$$

(The last equation means that, in the relevant topological algebra, the infinite sum on the right hand side is well defined and equals the left hand side.) Then, given a state ϕ, we have a probability measure p_ϕ on $(\Omega, \mathcal{F})$ given by

$$p_\phi(E) = \phi(\nu(E)) \quad \forall E \in \mathcal{F}. \tag{56}$$

The family $\{\nu(E), E \in \mathcal{F}\}$ will be called a *positive observable-valued measure* (PObVM) on $(\Omega, \mathcal{F})$. It is the abstract counterpart of the 'positive operator-valued measure' (POVM) employed in Hilbert space QM [18,21]. The objects $\nu(E)$ may be called *supmech events* (representing possible outcomes in a measurement situation); a state assigns probabilities to these events. Eq.(56) brings out quite transparently the relationship between the supmech expectation values and classical probabilities.

Denoting the algebraic dual of the superalgebra $\mathcal{A}$ by $\mathcal{A}^*$, an automorphism $\Phi : \mathcal{A} \to \mathcal{A}$ induces the transpose mapping $\tilde{\Phi} : \mathcal{A}^* \to \mathcal{A}^*$ such that

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \text{ or } < \tilde{\Phi}(\phi), A > = < \phi, \Phi(A) > \tag{57}$$

where the second alternative has employed the dual space pairing $<,>$. The mapping $\tilde{\Phi}$ (which is easily seen to be linear and bijective) maps states (which form a subset of $\mathcal{A}^*$) onto states. To see this, note that

(i) $\tilde{\Phi}(\phi)(A^*A) = \phi(\Phi(A^*A)) = \phi(\Phi(A)^*\Phi(A)) \geq 0$;

(ii)$[\tilde{\Phi}(\phi)](I) = \phi(\Phi(I)) = \phi(I) = 1$.

The linearity of $\tilde{\Phi}$ (as a mapping on the dual space of $\mathcal{A}$) ensures that it preserves convex combinations of states. In particular, it maps pure states onto pure states. We have, therefore, a bijective mapping $\tilde{\Phi} : \mathcal{S}_1(\mathcal{A}) \to \mathcal{S}_1(\mathcal{A})$.

When Φ is a canonical transformation, the condition $\Phi^*\omega = \omega$ gives, for $X, Y \in SDer(\mathcal{A})$,

$$\omega(X, Y) = (\Phi^*\omega)(X, Y) = \Phi^{-1}[\omega(\Phi_* X, \Phi_* Y)]$$

which gives

$$\Phi[\omega(X, Y)] = \omega(\Phi_* X, \Phi_* Y). \tag{58}$$

Taking expectation value of both sides of this equation in a state ϕ, we get

$$(\tilde{\Phi}\phi)[\omega(X, Y)] = \phi[\omega(\Phi_* X, \Phi_* Y)]. \tag{59}$$

The dependence on X,Y in this equation can be gotten rid of by defining ω_Φ by

$$\omega_\Phi(X, Y) = \omega(\Phi_* X, \Phi_* Y). \tag{60}$$

Eq(59) can now be written as

$$(\tilde{\Phi}\phi)[\omega(., .)] = \phi[\omega_\Phi(., .)]. \tag{61}$$

It is generally simpler to use Eq.(59). When Φ is an infinitesimal canonical transformation generated by $G \in \mathcal{A}$, we have

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \simeq \phi(A + \epsilon\{G, A\}). \tag{62}$$

Putting $\tilde{\Phi}(\phi) = \phi + \delta\phi$, we have

$$(\delta\phi)(A) = \epsilon\phi(\{G, A\}). \tag{63}$$

B. Dynamics

Dynamics is described by specifying an observable H, called the *Hamiltonian*; the evolution of the system is given in terms of the one-parameter family Φ_t of canonical transformations generated by H. (The parameter t is supposed to be an evolution parameter which need not always be the conventional time.) Writing $\Phi_t(A) = A(t)$ and recalling Eq.(40), we have the *Hamilton's equation* of supmech :

$$\frac{dA(t)}{dt} = Y_H[A(t)] = \{H, A(t)\}. \tag{64}$$

The triple $(\mathcal{A}, \omega, H)$ [or, more appropriately, the quadruple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$] will be called a *supmech Hamiltonian system*; it is the analogue of a classical Hamiltonian system (M, ω_{cl}, H_{cl})

[where (M, ω_{cl}) is a symplectic manifold and H_{cl}, the classical Hamiltonian (a smooth real-valued function on M); note that the specification of the symplectic manifold M serves to define both observables and pure states in classical mechanics]. As far as the evolution is concerned, the Hamiltonian is, as in the classical case, arbitrary up to the addition of a constant multiple of the identity element.

This is the analogue of the Heisenberg picture in traditional QM. An equivalent description is obtained by transferring the time dependence to states through the relation [see Eq.(57)]

$$< \phi(t), A >=< \phi, A(t) >$$

where $\phi(t) = \tilde{\Phi}_t(\phi)$. The mapping $\tilde{\Phi}_t$ satisfies the condition (61) which [with $\Phi = \Phi_t$] may be said to represent the canonicality of the evolution of states.

With $\Phi = \Phi_t$ and G = H, Eq.(63) gives the *Liouville equation* of supmech:

$$\frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \quad \text{or} \quad \frac{d\phi(t)}{dt}(.) = \phi(t)(\{H, .\}). \tag{65}$$

This is the analogue of the Schrödinger picture in traditional QM.

C. Equivalent descriptions; Symmetries and conservation laws

By a 'description' of a system, we shall mean specification of its triple $(\mathcal{A}, \omega, \mathcal{S}(\mathcal{A}))$. Two descriptions are said to be *equivalent* if they are related through a pair of isomorphisms $\Phi_1 : \mathcal{A} \to \mathcal{A}$ and $\Phi_2 : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$ such that the symplectic form and the expectation values are preserved :

$$\Phi_1^* \omega = \omega; \quad \Phi_2(\phi)[\Phi_1(A)] = \phi(A) \tag{66}$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$. The second equation above and Eq(57) imply tht we must have $\Phi_2 = (\tilde{\Phi}_1)^{-1}$. Two equivalent descriptions are, therefore, related through a canonical transformation on $\mathcal{A}$ and the corresponding inverse transpose transformation on the states. An infinitesimal transformation of this type generated by $G \in \mathcal{A}$ takes the form

$$\delta A = \epsilon\{G, A\}, \qquad (\delta\phi)(A) = -\epsilon\phi(\{G, A\}) \tag{67}$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$.

These transformations may be called symmetries of the formalism; they are the analogues of simultaneous unitary transformations on operators and state vectors in a Hilbert space preserving expectation values of operators. Symmetries of dynamics are the subclass of these which leave the Hamiltonian invariant:

$$\Phi_1(H) = H. \tag{68}$$

For an infinitesimal transformation generated by $G \in \mathcal{A}$, this equation gives

$$\{G, H\} = 0. \tag{69}$$

It now follows from the Hamilton equation (64) that (in the 'Heisenberg picture' evolution) G is a constant of motion. This is the situation familiar from classical and quantum mechanics: generators of symmetries of the Hamiltonian are conserved quantities and vice-versa.
Note. Noting that a symmetry operation is uniquely defined by any one of the two mappings Φ_1 and Φ_2, we can be flexible in the implementation of symmetry operations. It is often useful to implement them such that the symmetry operations act, in a single implementation, *either* on states *or* on observables, and the two actions are related the Heisenberg and Schrödinger picture evolutions above [see Eq.(57)]; we shall refer to this type of implementation as *unimodal*. In such an implementation, the second equation in (67) will not have a minus sign on the right.

For future reference, we define equivalence of supmech Hamiltonian systems. Two supmech Hamiltonian systems

$$(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H) \quad \text{and} \quad (\mathcal{A}', \mathcal{S}_1(\mathcal{A}'), \omega', H')$$

are said to be equivalent if they are related through a pair $\Phi = (\Phi_1, \Phi_2)$ of bijective mappings such that $\Phi_1 : (\mathcal{A}, \omega) \to (\mathcal{A}', \omega')$ is a symplectic mapping connecting the Hamiltonians [i.e. $\Phi_1(H) = H'$] and $\Phi_2 : \mathcal{S}_1(\mathcal{A}) \to \mathcal{S}_1(\mathcal{A}')$ such that $< \Phi_2(\phi), \Phi_1(A) >=< \phi, A >$.

D. Symplectic actions of Lie groups

In this subsection and the next section, we shall generally employ bosonic objects. The square brackets will, therefore, be commutators; the subscript – (minus) for the latter will be omitted.

The study of symplectic actions of Lie groups in supmech proceeds generally parallel to the classical case [85-87,81] and promises to be quite rich and rewarding. Here we shall present the essential developments mainly to provide background material for the next section.

Let G be a connected Lie group with Lie algebra $\mathcal{G}$. Elements of G, $\mathcal{G}$ and $\mathcal{G}^*$ (the dual space of $\mathcal{G}$) will be denoted, respectively, as g,h,.., $\xi, \eta, ..$ and $\lambda, \mu, ...$ The pairing between $\mathcal{G}^*$ and $\mathcal{G}$ will be denoted as $< ., . >$. Choosing a basis $\{\xi_a; a = 1, .., r\}$ in $\mathcal{G}$, we have the commutation relations $[\xi_a, \xi_b] = C^c_{ab}\xi_c$. The dual basis in $\mathcal{G}^*$ is denoted as $\{\lambda^a\}$ (so that $< \lambda^a, \xi_b >= \delta^a_b$). The action of G on $\mathcal{G}$ (adjoint representation) will be denoted as $Ad_g : \mathcal{G} \to \mathcal{G}$ and that on $\mathcal{G}^*$ (the coadjoint representation) by $Cad_g : \mathcal{G}^* \to \mathcal{G}^*$; the two are related as $< Cad_g\lambda, \xi >=< \lambda, Ad_{g^{-1}}\xi >$. With the bases chosen as above, the matrices in the two representations are related as $(Cad_g)_{ab} = (Ad_{g^{-1}})_{ba}$.

Recalling the mappings Φ_1 and Φ_2 of the previous subsection, the *symplectic action* of of G on a symplectic superalgebra $(\mathcal{A}, \omega)$ is given by the assignment, to each $g \in G$, a symplectic mapping (canonical transformation) $\Phi_1(g) : \mathcal{A} \to \mathcal{A}$ which is a group action [which means that $\Phi_1(g)\Phi_1(h) = \Phi_1(gh)$ and $\Phi_1(e) = id_{\mathcal{A}}$ in obvious notation]. The action on the states is given by the mappings $\Phi_2(g) = [\tilde{\Phi}_1(g)]^{-1}$.

A one-parameter subgroup g(t) of G generated by $\xi \in \mathcal{G}$ induces a locally Hamiltonian derivation $Z_\xi \in SDer(\mathcal{A})$ as the generator of the one-parameter family $\Phi_1(g(t))$ of canonical transformations of $\mathcal{A}$. The correspondence $\xi \to Z_\xi$ is a Lie algebra homomorphism : $Z_{[\xi,\eta]} = [Z_\xi, Z_\eta]$. The G-action is said to be *hamiltonian* if these derivations are Hamiltonian, i.e. for

each $\xi \in \mathcal{G}, Z_\xi = Y_{h_\xi}$ for some $h_\xi \in \mathcal{A}$ (called the *hamiltonian* corresponding to ξ). These hamiltonians are arbitrary upto addition of multiples of the unit element. This arbitrariness can be somewhat reduced by insisting that h_ξ be linear in ξ.(This can be achieved by first defining the hamiltonians for the members of a basis in $\mathcal{G}$ and then defining them for general elements as appropriate linear combinations of these.) We shall always assume this linearity.

A hamiltonian G-action satisfying the additional condition

$$\{h_\xi, h_\eta\} = h_{[\xi,\eta]} \text{ for all } \xi, \eta \in \mathcal{G} \tag{70}$$

is called a *Poisson action*. The hamiltonians of a Poisson action have the following equivariance property :

$$\Phi_1(g)(h_\xi) = h_{Ad_g(\xi)}. \tag{71}$$

Since G is connected, it is adequate to verify this relation for infinitesimal group actions. Denoting by g(t) the one-parameter group generated by $\eta \in \mathcal{G}$, we have, for small t,

$$\Phi_1(g(t))(h_\xi) \simeq h_\xi + t\{h_\eta, h_\xi\} = h_\xi + t h_{[\eta,\xi]} = h_{\xi + t[\eta,\xi]} \simeq h_{Ad_{g(t)}\xi}$$

completing the verification.

A Poisson action is not always admissible. The obstruction to such an action is determined by the objects

$$\alpha(\xi, \eta) = \{h_\xi, h_\eta\} - h_{[\xi,\eta]} \tag{72}$$

which are easily seen to have vanishing Hamiltonian derivations [i.e. $Y_{\alpha(\xi,\eta)} = 0$] and hence vanishing Poisson brackets with all elements of $\mathcal{A}$. [This last condition defines the so-called *neutral elements* [85] of the Lie algebra$(\mathcal{A}, \{,\})$. They clearly form a complex vector space which will be finite dimensional in in the situations we shall encounter.] We also have

$$\alpha([\xi,\eta],\zeta) + \alpha([\eta,\zeta],\xi) + \alpha([\zeta,\xi],\eta) = 0;$$

i.e. α is a 2-cocycle of $\mathcal{G}$ taking values in the space $\mathcal{N}$ of the neutral elements of $\mathcal{A}$. (The derivation [81] of this result in classical mechanics employs only the standard properties of PBs and remains valid in supmech.) A redefinition of the hamiltonians $h_\xi \to h'_\xi = h_\xi + k_\xi I$ (where the scalars k_ξ are linear in ξ) changes α by a coboundary term:

$$\alpha'(\xi, \eta) = \alpha(\xi, \eta) - k_{[\xi,\eta]}$$

showing that the obstruction is characterized by a cohomology class of $\mathcal{G}$. A necessary and sufficient condition for the admissibility of Poisson action of G on $\mathcal{A}$ is that it should be possible to transform away all the obstruction 2-cocycles by redefining the hamiltonians, or, equivalently, $H^2(\mathcal{G}, \mathcal{N}) = 0$.

As in classical symplectic mechanics [85,88], Hamiltonian group actions (more generally, the Lie algebra actions) with nontrivial neutral elements can be treated as Poisson actions of a (Lie

group with a) larger Lie algebra $\hat{\mathcal{G}}$ obtained by adding, to a basis of $\mathcal{G}$, elements corresponding to a basis of the space $\mathcal{N}$ of neutral elements mentioned above. Let $\eta_r(.,.)(r = 1, .., m)$ be a basis of $\mathcal{N}$. We add extra generators M_r to the basis $\{\xi_a\}$ and take the commutation relations of the larger Lie algebra $\hat{\mathcal{G}}$ as

$$[\xi_a, \xi_b] = C^c_{ab}\xi_c + \sum_{r=1}^{m} \eta_r(\xi_a, \xi_b)M_r; \quad [\xi_a, M_r] \quad = \quad 0 \quad = \quad [M_r, M_s]. \tag{73}$$

The simply connected Lie group $\hat{G}$ with the Lie algebra $\hat{\mathcal{G}}$ is called the *projective group* [88] of G; it is generally a central extension of the universal covering group $\tilde{G}$ of G.

Momentum map. In classical mechanics, given a Poisson action of a connected Lie group G on a symplectic manifold (M, ω_{cl}) [with hamiltonians/comoments $h^{(cl)}_\xi \in C^\infty(M)$], a useful construction is the so-called *momentum map* [89,86,87] $P : M \to \mathcal{G}^*$ given by

$$< P(x), \xi > \; = \; h^{(cl)}_\xi(x) \;\; \forall x \in M \text{ and } \;\; \xi \in \mathcal{G}. \tag{74}$$

This map relates the symplectic action Φ_g of G on M ($\Phi_g : M \to M, \Phi^*_g\omega_{cl} = \omega_{cl} \; \forall g \in G$) and the transposed adjoint action on $\mathcal{G}^*$ through the equivariance property

$$P(\Phi_g(x)) = Ad^*_g(P(x)) \; \forall x \in M \text{ and } \;\; g \in G. \tag{75}$$

Noting that points of M are pure states of the algebra $\mathcal{A}_{cl} = C^\infty(M)$, the map P may be considered as the restriction to M of the dual/transpose $\tilde{h}^{(cl)} : \mathcal{A}^*_{cl} \to \mathcal{G}^*$ of the linear map $h^{(cl)} : \mathcal{G} \to \mathcal{A}_{cl}$ [given by $h^{(cl)}(\xi) = h^{(cl)}_\xi$]:

$$< \tilde{h}^{(cl)}(u), \xi > = < u, h^{(cl)}(\xi) > \;\; \forall \; u \in \mathcal{A}^*_{cl} \text{ and } \;\; \xi \in \mathcal{G}. \tag{76}$$

The analogue of M in supmech is $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{A})$. Defining $h : \mathcal{G} \to \mathcal{A}$ by $h(\xi) = h_\xi$, the analogue of the momentum map in supmech is the mapping $\tilde{h} : \mathcal{S}_1 \to \mathcal{G}^*$ (considered as the restriction to $\mathcal{S}_1$ of the mapping $\tilde{h} : \mathcal{A}^* \to \mathcal{G}^*$) given by

$$< \tilde{h}(\phi), \xi > = < \phi, h(\xi) > = < \phi, h_\xi > . \tag{77}$$

Recalling the symplectic mappings Φ_1 and Φ_2 and Eq.(71), we have

$$\begin{aligned} < \tilde{h}(\Phi_2(g)\phi), \xi > \quad &= \quad < \Phi_2(g)\phi, h_\xi > = < \phi, \Phi_1(g^{-1})(h_\xi) > = < \phi, h_{Ad_{g^{-1}}(\xi)} > \\ &= \quad < \phi, h(Ad_{g^{-1}}(\xi)) > = < Cad_g(\tilde{h}(\phi)), \xi > \end{aligned}$$

giving finally

$$\tilde{h}(\Phi_2(g)\phi) = Cad_g(\tilde{h}(\phi)) \tag{78}$$

which is the supmech analogue of Eq.(75). [Note. In Eq.(78),the co-adjoint (and not the transposed adjoint) action appears on the right because $\Phi_2(g)$ is inverse transpose (and not transpose) of $\Phi_1(g)$.]

E. Generalized symplectic structures and Hamiltonian systems

The generalization of the DVNCG scheme introduced in section III E can be employed to obtain a corresponding generalization of the supmech formalism. One picks up a distinguished Lie sub-superalgebra $\mathcal{X}$ of $SDer(\mathcal{A})$ and restricts the superderivations of $\mathcal{A}$ in all definitions and constructions to those in $\mathcal{X}$. Thus, a symplectic superalgebra $(\mathcal{A}, \omega)$ is now replaced by a *generalized symplectic superalgebra* $(\mathcal{A}, \mathcal{X}, \omega)$ and a symplectic mappings $\Phi : (\mathcal{A}, \mathcal{X}, \alpha) \rightarrow (\mathcal{B}, \mathcal{Y}, \beta)$ is restricted to a superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\Phi_* : \mathcal{X} \rightarrow \mathcal{Y}$ is a Lie-superalgebra- isomorphism and $\Phi^*\beta = \alpha$. A supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ is now replace by a *generalized supmech Hamiltonian sytem* $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \mathcal{X}, \omega, H)$. In section VII E, we shall employ the pairs ($\mathcal{A}$, $\mathcal{X}$) with $\mathcal{X}$= ISDer($\mathcal{A}$) to define quantum symplectic structure on superalgebras admitting outer as well as inner superderivations.

Note. In Ref.[2], the formalism in section IV (of that paper) was developed right from the beginning in terms of the pairs ($\mathcal{A}$, $\mathcal{X}$) (called 'algebraic differential systems' there; we have chosen to dispense with this nomenclature to avoid confusion of the term 'differential system' with its use elsewhere in mathematics literature). This has the advantage of extra generality; however, noting that this generality is needed only at very few places and its use everywhere would make the formalism unnecessarily more complicated, the author has not opted for it in the present work.

F. Augmented symplectics including time; the generalized Poincaré-Cartan form

We shall now augment the kinematic framework of supmech by including time and obtain the non-commutative analogues of the Poincaré-Cartan form and the symplectic version of Noether's theorem [89].

For a system S with associated symplectic superalgebra $(\mathcal{A}, \omega)$ we construct the *extended system algebra* $\mathcal{A}^e = C^\infty(R) \otimes \mathcal{A}$ (where the real line R is the carrier space of the 'time' t) whose elements are finite sums $\sum_i f_i \otimes A_i$ (with $f_i \in C^\infty(R) \equiv \mathcal{A}_0$) which may be written as $\sum_i f_i A_i$. This algebra is the analogue of the algebra of functions on the evolution space of Souriau [89] (the Cartesian product of the time axis and the phase space — often referred to as the phase bundle).The superscript e in $\mathcal{A}^e$, may, therefore, also be taken to refer to 'evolution'.

Derivations on $\mathcal{A}_0$ are of the form $g(t)\frac{d}{dt}$ and one-forms of the form h(t)dt where g and h are smooth functions; there are no nonzero higher order forms. We have, of course, $dt(\frac{d}{dt}) = 1$.

A (super-)derivation D_1 on $\mathcal{A}_0$ and D_2 on $\mathcal{A}$ extend trivially to (super-)derivations on $\mathcal{A}^e$ as $D_1 \otimes id_{\mathcal{A}}$ and $id_{\mathcal{A}_0} \otimes D_2$ respectively (where $id_{\mathcal{A}}$ is the identity mapping on $\mathcal{A}$); these trivial extensions may be informally denoted as D_1 and D_2. With $f \otimes A$ written as fA, we can write $D_1(fA) = (D_1 f)A$ and $D_2(fA) = f(D_2 A)$.

The mapping $\Xi : \mathcal{A} \rightarrow \mathcal{A}^e$ given by $\Xi(A) = 1 \otimes A (= A)$ is an isomorphism of the algebra $\mathcal{A}$ onto the subalgebra $\tilde{\mathcal{A}} \equiv 1 \otimes \mathcal{A}$ of $\mathcal{A}^e$ and can be employed to pull back the differential forms on $\mathcal{A}$ to those on $\tilde{\mathcal{A}}$. We write $(\Xi^{-1})^*(\omega) = \tilde{\omega}$ and extend this form on $\tilde{\mathcal{A}}$ to one on $\mathcal{A}^e$ by

defining

$$\tilde{\omega}(\frac{d}{dt}, X) = 0 \quad \forall X \in SDer(\mathcal{A}).$$

We shall generally skip the tilde. Similarly, we may extend the one-form dt on $\mathcal{A}_0$ to one on $\mathcal{A}^e$ by defining (dt)(X) =0 for all $X \in SDer(\mathcal{A})$.

The symplectic structure ω on $\mathcal{A}$ induces, on $\mathcal{A}^e$, a generalized symplectic structure (of the type introduced in section IV E) with the distinguished Lie sub-superalgebra $\mathcal{X}$ of $Sder(\mathcal{A}^e)$ taken to be the one consisting of the objects $\{id_{\mathcal{A}_0} \otimes D; D \in SDer(\mathcal{A})\}$ which constitute a Lie sub-superalgebra of $SDer(\mathcal{A}^e)$ isomorphic to $SDer(\mathcal{A})$, thus giving a generalized symplectic superalgebra $(\mathcal{A}^e, \mathcal{X}, \tilde{\omega})$. The corresponding PBs on $\mathcal{A}^e$ are trivial extensions of those on $\mathcal{A}$ obtained by treating the 'time' t as an external parameter; this amounts to extending it by $\mathcal{A}_0$-linearity :

$$\{fA + gB, hC\}_{\mathcal{A}^e} = fh\{A, C\}_{\mathcal{A}} + gh\{B, C\}_{\mathcal{A}}$$

where, for clarity, we have put subscripts on the PBs to indicate the underlying superalgebras. We shall often drop these subscripts; the underlying (super-)algebra will be clear from the context.

To describe dynamics in $\mathcal{A}^e$, we extend the one-parameter family Φ_t of canonical transformations on $\mathcal{A}$ generated by a Hamiltonian $H \in \mathcal{A}$ to a one-parameter family Φ^e_t of tansformations on $\mathcal{A}^e$ (which are 'canonical' in a certain sense, as we shall see below) given by

$$\Phi^e_t(fA) \equiv (fA)(t) = f(t)A(t) \equiv (\Phi^{(0)}_t f)\Phi_t(A)$$

where $\Phi^{(0)}_t$ is the one-parameter group of translations on $\mathcal{A}_0$ generated by the derivation $\frac{d}{dt}$. An infinitesimal transformation under the evolution Φ^e_t is given by

$$\begin{aligned} \delta(fA)(t) &\equiv (fA)(t + \delta t) - (fA)(t) \\ &= [\frac{df}{dt}A + f\{H, A\}_{\mathcal{A}}]\delta t \equiv \hat{Y}_H(fA)\delta t \end{aligned}$$

where

$$\hat{Y}_H = \frac{\partial}{\partial t} + \tilde{Y}_H. \tag{79}$$

Here $\frac{\partial}{\partial t}$ is the derivation on $\mathcal{A}^e$ corresponding to the derivation $\frac{d}{dt}$ on $\mathcal{A}_0$ and

$$\tilde{Y}_H = \{H, .\}_{\mathcal{A}^e}. \qquad (*)$$

Note that
(i) $dt(\hat{Y}_H) = 1$;
(ii) the right hand side of the equation (*) remains well defined if $H \in \mathcal{A}^e$ ('time dependent' Hamiltonian). Henceforth, in various formulas in this subsection, H will be understood to be an element of $\mathcal{A}^e$.

The obvious generalization of the supmech Hamilton equation (64) to $\mathcal{A}^e$ is the equation

$$\frac{dF(t)}{dt} = \hat{Y}_H F(t) = \frac{\partial F(t)}{\partial t} + \{H(t), F(t)\}. \tag{80}$$

We next consider an object in $\mathcal{A}^e$ which contains complete information about the symplectic structure *and* dynamics [i.e. about $\tilde{\omega}$ and H (up to an additive constant multiple of I)] and is canonically determined by these objects. It is the 2-form

$$\Omega = \tilde{\omega} + dt \wedge dH \tag{81}$$

which is 'obviously' closed. [To have a formal proof, apply Eq.(133) below with $\Omega = 1 \otimes \omega + d_1 t \wedge d_2 H$ where the exterior product is considered as the appropriate linear combination of tensor product terms.] If the symplectic structure on $\mathcal{A}$ is exact (with $\omega = d\theta$), we have ('obviously') $\Omega = d\Theta$ where

$$\Theta = \tilde{\theta} - H dt \tag{82}$$

is the supmech avatar of the Poincaré-Cartan form in classical mechanics. [Again, for a formal derivation, use Eq.(133) with $\Theta = 1 \otimes \theta - dt \otimes H$.]

The closed form Ω is generally not non-degenerate. It defines what may be called a *presymplectic structure* [89] on $\mathcal{A}^e$. In fact, we have here the noncommutative analogue of a special type of presymplectic structure called *contact structure* [90,91]; it may be called the Poincaré-Cartan contact structure. We shall, however, not attempt a formal development of noncommutative contact structures here.

A *symplectic action* of a Lie group G on the presymplectic space $(\mathcal{A}^e, \Omega)$ is the assignment, to every $g \in G$, an automorphism $\Phi(g)$ of the superalgebra $\mathcal{A}^e$ having the usual group action properties and such that $\Phi(g)^* \Omega = \Omega$. This implies, as in section IV D, that, to every element ξ of the Lie algebra $\mathcal{G}$ of G, corresponds a derivation Z_ξ such that $L_{Z_\xi} \Omega = 0$ which, in view of the condition $d\Omega = 0$, implies

$$d(i_{Z_\xi} \Omega) = 0. \tag{83}$$

We shall now verify that the one-parameter family $\Phi_t^{(e)}$ of transformations on $\mathcal{A}^e$ is symplectic/canonical. For this, it is adequate to verify that Eq.(83) holds with $Z_\xi = \hat{Y}_H$. We have, in fact, the stronger relation

$$i_{\hat{Y}_H} \Omega = 0. \tag{84}$$

Indeed

$$\begin{aligned} i_{\hat{Y}_H} \Omega &= i_{\partial/\partial t} \Omega + i_{\tilde{Y}_H} \Omega \\ &= i_{\partial/\partial t}(dt \wedge dH) + i_{\tilde{Y}_H} \tilde{\omega} + i_{\tilde{Y}_H}(dt \wedge dH) \\ &= dH - dH - i_{\tilde{Y}_H}(dH) dt \\ &= [i_{\tilde{Y}_H}(i_{\tilde{Y}_H} \tilde{\omega})] dt = 0. \end{aligned}$$

The equation in note (i) above and Eq.(84) are analogous to the properties of the 'characteristic vector field' of a contact structure. The derivation $\hat{Y}_H$ may, therefore, be called the *characteristic derivation* of the Poincaré-Cartan contact structure.

A symplectic G-action (in the present context) is said to be *hamiltonian* if the 1-forms $i_{Z_\xi}\Omega$ are exact, i.e. to each $\xi \in \mathcal{G}$, corresponds a 'hamiltonian' $\hat{h}_\xi \in \mathcal{A}^e$ (unique upto an additive constant) such that

$$i_{Z_\xi}\Omega = -d\hat{h}_\xi. \tag{85}$$

These 'hamiltonians' (*Noether invariants*) are constants of motion :

$$\begin{aligned} \frac{d\hat{h}_\xi(t)}{dt} &= \hat{Y}_H(\hat{h}_\xi(t)) = (d\hat{h}_\xi)(\hat{Y}_H)(t) \\ &= -(i_{Z_\xi}\Omega)(\hat{Y}_H)(t) = 0 \end{aligned} \tag{86}$$

where, in the last step, Eq.(84) has been used. This is the supmech analogue of the symplectic version of Noether's theorem. For some concrete examples of Noether invariants, see section V B.

G. Systems with configuration space; lacalizability

We shall now consider the class of systems each of which has a configuration space (say, M) associated with it and it is meaningful to ask questions about the localization of the system in subsets of M. To start with, we shall take M to be a topological space and restrict the permitted domains of localization to belong to B(M), the family of Borel subsets of M.

Some good references containing detailed treatment of localization in conventional approaches are Ref.[92,93,47,94]. We shall follow a relatively more economical path exploiting some of the constructions described above.

We shall say that a system S [with associated symplectic superalgebra $(\mathcal{A}, \omega)$] is *localizable* in M if, we have a positive observable-valued measure (as defined in section IV A) on the measurable space (M,B(M)), which means that, corresponding to every subset $D \in B(M)$, there is a positive observable P(D) in $\mathcal{A}$ satisfying the three conditions
(i) $P(\emptyset) = 0$; (ii) P(M) = I;
(iii)for any countable family of mutually disjoint sets $D_i \in B(M)$,

$$P(\cup_i D_i) = \sum_i P(D_i). \tag{87}$$

For such a system, we can associate, with any state ϕ, a probability measure μ_ϕ on the measurable space $(M, B(M))$ defined by [see Eq.(56)]

$$\mu_\phi(D) = \phi(P(D)), \tag{88}$$

making the triple $(M, B(M), \mu_\phi)$ a probability space. The quantity $\mu_\phi(D)$ is to be interpreted as the probability of the system, given in the state ϕ, being found (on observation/measurement) in the domain D.

Generally it is of interest to consider localizations having suitable invariance properties under a transformation group G. Typically G is a topological group with continuous action on M assigning, to each $g \in G$, a bijection $T_g : M \to M$ such that, in obvious notation, $T_g T_{g'} = T_{gg'}$ and $T_e = id_M$; it also has a symplectic action on $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ given by the mappings $\Phi_1(g)$ and $\Phi_2(g)$ introduced in section IV D. The localization on M described above will be called *G-covariant* (or, loosely, G-invariant) if

$$\Phi_1(g)(P(D)) = P(T_g(D)) \ \ \forall g \in G \text{ and } D \in B(M). \tag{89}$$

In most practical applications, M is a manifold and G is a Lie group with smooth action on M and a Poisson action on the symplectic superalgebra $(\mathcal{A}, \omega)$.

In Hilbert space QM, the problem of G-covariant localization is traditionally formulated in terms of the so-called 'systems of imprimitivity' [95,47,93]. We are operating in the more general algebraic setting trying to exploit the machinery of noncommutative symplectics developed above. Clearly, there is considerable scope for mathematical developments in this context parallel to those relating to systems of imprimitivity. We shall, however, restrict ourselves to some essential developments relevant to the treatment of localizable elementary systems (massive particles) later.

We shall be mostly concerned with $M = R^n$ (equipped with the Euclidean metric). In this case, one can consider averages of the form (denoting the natural coordinates on R^n by x_j)

$$\int_{R^n} x_j d\mu_\phi(x), \quad j = 1, ..., n. \tag{90}$$

It is natural to introduce *position/configuration observables* X_j such that the quantity (90) is $\phi(X_j)$. Let E_n denote the Euclidean group in n dimensions and let $p_j, m_{jk} (= -m_{kj})$ be its generators satisfying the usual commutation relations. We shall say that a system S with configuration space R^n has *concrete Euclidean-covariant localization* if it is localizable as above and

(i) it has position observables $X_j \in \mathcal{A}$ such that the above mentioned relation holds :

$$\phi(X_j) \ \ = \ \ \int_{R^n} x_j d\mu_\phi(x); \tag{91}$$

(The term 'concrete' is understood to imply this condition.)

(ii) the group E_n has a Poisson action on $\mathcal{A}$ so that we have the hamiltonians P_j, M_{jk} associated with p_j, m_{jk} such that

$$\{P_j, P_k\} = 0, \quad \{M_{jk}, P_l\} = \delta_{jl} P_k - \delta_{kl} P_j$$
$$\{M_{jk}, M_{pq}\} = \delta_{jp} M_{kq} - \delta_{kp} M_{jq} - \delta_{jq} M_{kp} + \delta_{kq} M_{jp}; \tag{92}$$

(iii) the covariance condition (89) holds with the Euclidean action on R^n given by

$$T_{(R,a)} x = Rx + a, \quad R \in SO(n), \ a \in R^n. \tag{93}$$

Equations (91) and (89) for infinitesimal Euclidean transformations and Eq.(67) then give the analogues of the classical canonical PBs of X_js with the Euclidean generators :

$$\{P_j, X_k\} = \delta_{jk} I, \quad \{M_{jk}, X_l\} = \delta_{jl} X_k - \delta_{kl} X_j. \tag{94}$$

Proof. Using Eq.(91) with ϕ replaced by $\phi' = \Phi_2(g)(\phi)$, we have

$$\phi'(X_j) = \int x_j d\mu_{\phi'}(x) = \int x_j d\mu_\phi(x') = \int (x'_j - \delta x_j) d\mu_\phi(x') \tag{95}$$

where $x' \equiv T_g(x) = x + \delta x$ and we have used Eq.(89) with D as a single point set (these are included in $B(R^n)$) to write $d\mu_{\phi'}(x) = d\mu_\phi(x')$. Writing $\phi' = \phi + \delta\phi$ and taking T_g to be a general infinitesimal transformation generated by $\epsilon\xi = \epsilon^a \xi_a$, we have

$$\epsilon\phi(\{h_\xi, X_j\}) = \int_{R^n} \delta x_j d\mu_\phi(x). \tag{96}$$

For translations, with $\xi = p_k, h_{p_k} = P_k, \delta x_j = \epsilon\delta_{jk}$, Eq.(96) gives

$$\phi(\{P_k, X_j\}) = \delta_{jk} = \delta_{jk}\phi(I).$$

Since this holds for all $\phi \in \mathcal{S}(\mathcal{A})$, we have the first of the equations (94). The second equation is similarly obtained by taking, in obvious notation, $\epsilon\xi = \frac{1}{2}\epsilon_{jk} m_{jk}$ and

$$\delta x_l = \epsilon_{lk} x_k = \epsilon_{jk}\delta_{jl} x_k = \frac{1}{2}\epsilon_{jk}(\delta_{jl} x_k - \delta_{kl} x_j). \ \square$$

The hamiltonians P_j and M_{jk} will be referred to as the *momentum* and *angular momentum* observables of the system S. It should be noted that the PBs obtained above do not include the expected relations $\{X_j, X_k\} = 0$; these relations, as we shall see in the following section, come from the relativity group. [Recall that, in the treatments of localalization based on systems of imprimitivity, the commutators $[X_j, X_k] = 0$ appear because there the analogues of the objects P(D) are assumed to be projection operators satisfying the relation $P(D)P(D') = P(D \cap D')(= P(D')P(D))$. In our more general approach, we have no basis for making such assumptions.]

V. RELATIVITY GROUPS, ELEMENTARY SYSTEMS AND FUNDAMENTAL OBSERVABLES

Having presented the general formalism of supmech, we now proceed to take the first steps towards the treatment of concrete systems. We start with the simplest ones : particles. In this section, we take up the question of the definition of a particle and the fundamental observables relating to the characterization/labelling and kinematics of a particle. Relativity group will be seen to play a very important role in this context.

A. General considerations about relativity groups and elementary systems

A particle is basically an irreducible entity (in the sense that it cannot be represented as the composite of more than one identifiable entities) localized in what we traditionally call 'space'

and the description of its dynamics involves 'time'. We must, therefore, introduce the concepts of space and time or, more generally, space-time before we talk about particles.

In the following developments, *space-time* will be understood to be a (3+1)- dimensional differentiable manifold equipped with a suitable metric to define spatial distances and time-intervals.[The breakup (3+1) means that one of the four dimensions is in some way distinguished from the other three. Details about the metric will be given only when needed.] A *reference frame* is an atlas providing a coordinatization of the spacetime points. *Observers* are supposedly intelligent beings employing reference frames for doing concrete physics; they will be understood to be in one-to-one correspondence with reference frames.

Observables of systems localized in space are generally observer-dependent. This observer dependence is systematically taken into consideration by adopting a *relativity scheme* which incorporates (i) specification of the geometry of spacetime, (ii) selection of a class of reference frames to to be treated as distinguished (all members of the chosen class to be treated as physically equivalent) and (iii) transformation laws between coordinatizations of different members of the chosen class (these transformations constitute a group called the *relativity group* of the scheme).

Assuming a fixed background spacetime M, we shall assume the relativity group to be a connected Lie group G_0 (with Lie algebra $\mathcal{G}_0$) acting as a transformation group on M. For concrete applications, we shall take G_0 to be the Galilean group and the Poincaré group (the inhomogeneous Lorentz group) in the schemes of Galilean relativity and special relativity respectively. Both these groups have the one-parameter group $\mathcal{T}$ of time translations as a subgroup.

Treatment of kinematics and dynamics of a system in accordance with a relativity scheme involves the action of the relevant relativity group on the symplectic algebra $(\mathcal{A}, \omega)$ of the system. To exploit the availability of a symplectic framework, we would like this action to be a Poisson action so that we can associate observables with the infinitesimal generators of the relativity group. As one of the relativity groups (the Galilean group) to be considered for concrete applications does not admit Poisson action, we shall employ its projective group $\hat{G}_0$ defined in section IV D. In fact, we shall operate directly with what we call the *effective relativity group* (denoted as $\hat{G}_0$) which is the universal covering group $\tilde{G}_0$ of the relativity group G_0 if it admits Poisson action and its projective group if it does not.

We may formally state, in the sub-domain of supmech covering theories admitting a background space-time, the 'principle of relativity' as follows :

(i) There is a preferred class of reference frames whose space-time coordinatisations are related through the action of a connected Lie group G_0.
(ii) For a system with the system algebra $\mathcal{A}$, the effective relativity group $\hat{G}_0$ has a Poisson action on the symplectic algebra $(\mathcal{A}, \omega)$ [or the generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega)$ in appropriate situations].
(iii) All reference frames in the chosen class are physically equivalent in the sense that the fundamental equations of the theory are covariant with respect to the G_0-transformations of

the relevant variables.

We shall call such a scheme G_0-*relativity* and systems covered by it G_0-*relativistic*.

Heisenberg and Schrödinger pictures of dynamics corresponding to two observers O and O′ may be related through the symplectic action of $\hat{G}_0$ by following the strategy of Ref.[85] exploiting the fact that $\hat{G}_0$ has $\mathcal{T}$ as a subgroup. Showing the observer dependence of the algebra elements explicitly, the two Heisenberg picture descriptions A(O,t) and A(O′,t′)of an element A of $\mathcal{A}$ can be related through the sequence (assuming a common zero of time for the two observers)

$$A(O,t) \longrightarrow A(O,0) \longrightarrow A(O',0) \longrightarrow A(O',t')$$

where the first and the last steps involve the operations of time translations in the two frames.We shall be concerned only with the symplectic action of $\hat{G}_0$ involved in the middle step. A similar strategy can be adopted for the Schrödinger picture. Detailed treatments of the relativistic Heisenberg and Schrödinger pictures in the classical hamiltonian formalism may be found in Ref.[85].

Construction of Noether invariants, on the other hand, involves explicit consideration of the transformation of the time variable. The formalism of section IV F has obvious limitations in this regard because time was treated as an external parameter in the Poisson brackets employed there. We shall, therefore, construct the Noether invariants only for the Galilean group where the only admitted transformations of the time variable are translations.

To formalize the notion of a particle as an irreducible entity, Wigner [96] introduced the concept of an 'elementary system' as a quantum system whose Hilbert space carries a projective unitary irreducible representation of the Poincaré group. The basic idea is that the state space of an elementary system should not admit a decomposition into more than one invariant (under the action of the relevant relativity group) subspaces. Following this idea, elementary systems in classical mechanics [85,88a] have been defined in terms of a transitive action of the relativity group on the phase space of the system. Alonso [88a] gave a unified treatment of classical and quantum elementary systems by treating them as special cases of (irreducible/transitive) kinematical action of the relativity groups (called 'invariance groups' in that work) on the state space of a dynamical system.

In this section, we shall treat elementary systems in the framework of supmech. Traditional classical and quantum elementary systems will be seen as special cases of these. This treatment goes a step further than that of Alonso in that the unification is achieved in a single *symplectic* framework.

A system S having associated with it the symplectic triple $(\mathcal{A}, \mathcal{S}_1, \omega)$ is called an *elementary system* in G_0-relativity if it is a G_0-relativistic system such that the action of $\hat{G}_0$ on the space $\mathcal{S}_1$ of its pure states is transitive. Formally, an elementary system may be represented as a collection $\mathcal{E} = (G_0, \mathcal{A}, \mathcal{S}_1, \omega, \Phi)$ where $\Phi = (\Phi_1, \Phi_2)$ are mappings as in section IV D implementing the $\hat{G}_0$-actions — Φ_1 describing a Poisson action on $(\mathcal{A}, \omega)$ and Φ_2 a transitive action on $\mathcal{S}_1$.

Note that, unlike the prevalent practice in the literature on classical and quantum elementary systems, our definition directly employs the action of the effective relativity group $\hat{G}_0$. This not only simplifies the theoretical treatment, but also makes sense from the point of view of concrete everyday physics. For example, in Newtonian mechanics, which is governed by the Galilean relativity scheme, one cannot proceed an inch before introducing the concept of mass and this comes directly (along with other kinematic variables), as we shall see below, when working with the projective group of the Galilean group.

Let ξ_a (a = 1,..,r) be a basis in the Lie algebra $\hat{\mathcal{G}}_0$ of $\hat{G}_0$ satisfying the commutation relations as in section IV D. The admissibility of Poisson action of $\hat{G}_0$ on $\mathcal{A}$ implies that, corresponding to the generators ξ_a, we have the hamiltonians $h_a \equiv h_{\xi_a}$ in $\mathcal{A}$ satisfying the PB relations

$$\{h_a, h_b\} = C^c_{ab} h_c. \tag{97}$$

Recalling Eq.(78), the condition of transitive action on $\mathcal{S}_1$ implies that the $\tilde{h}$-images of pure states of an elementary system are coadjoint orbits in $\hat{\mathcal{G}}_0^*$.

In classical mechanics, one has an isomorphism between the symplectic structures on the symplectic manifolds of elementary systems and those on the coadjoint orbits. In our case, the state spaces of elementary systems and coadjoint orbits of relativity groups are generally spaces of different types and the question of an isomorphism does not arise. We can, however, use Eq.(78) to obtain useful information about the transformation properties of the quantities h_a under the $\hat{G}_0$- action. Recalling the notations in section IV D, writing $Cad_g[\tilde{h}(\phi)] = u_a(g)\lambda^a$, we have

$$u_a(g) =< Cad_g[\tilde{h}(\phi)], \xi_a >=< \tilde{h}[\Phi_2(g)(\phi)], \xi_a >=< \phi, \Phi_1(g^{-1})h_a > \tag{98}$$

showing that the transformation properties of hamiltonians h_a are directly related to those of the corresponding coordinates (with respect to the dual basis) of points on the relevant co-adjoint orbit. This is adequate to enable us to to use the descriptions of the relevant co-adjoint actions in Ref.[88a] and draw parallel conclusions.

We shall adopt the following strategy :

(i) Given a relativity scheme, use the Poisson action of the corresponding effective relativity group $\hat{G}_0$ on the symplectic superalgebra of an elementary system to obtain the corresponding hamiltonians and their PBs [Eq.(97)]. These PBs are clearly the same for all elementary systems of the group.

(ii) Use these PBs to identify some *fundamental observables* [i.e. those which cannot be obtained from other observables (through algebraic relations or PBs)]. These include observables (like mass) that Poisson-commute with all h_as and the momentum observables (the group of space translations being a subgroup of both the relativity groups we consider).

(iii) Determine the transformation laws of h_as under finite transformations of $\hat{G}_0$ following the relevant developments in Ref.[85,88a]. Use these transformation laws to identify the $\hat{G}_0$-

invariants and some other fundamental observables (the latter will be configuration and spin observables).

The transitive action of $\hat{G}_0$ on $\mathcal{S}_1$ implies that the expectation value of a $\hat{G}_0$-invariant observable is the same in every pure state (hence in every state). To see this, let Q be such an observable and ϕ_1, ϕ_2 two pure states such that $\Phi_2(g)(\phi_1) = \phi_2$ for some $g \in \hat{G}_0$. We have

$$< \phi_2, Q > = < \Phi_2(g)(\phi_1), Q > = < \phi_1, \Phi_1(g^{-1})(Q) > = < \phi_1, Q >$$

as desired. Denoting this common expectation value of Q by q (we shall call it the value of Q for the system), we have, by the CC condition, Q = qI. The values of invariant observables characterize (or serve to label) an elementary system.

(iv) The system algebra $\mathcal{A}$ for an elementary system is to be taken as the (topological completion of) the one generated by the fundamental observables and the identity element.

(v) Obtain (to the extent possible) the general form of the Hamiltonian as a function of the fundamental observables as dictated by the PB relations (97).
(vi) (For the Galilean group) use the formalism of section IV F to consider the action of $\hat{G}_0$ on the presymplectic space $(\mathcal{A}^e, \Omega)$ and, noting that this action satisfies Eq.(85), identify the Noether invariants related to $\hat{G}_0$.

We shall now obtain an equation that will be useful for this last job. Let $\xi \in \hat{\mathcal{G}}_0$ generate an infinitesimal transformation giving $\delta t = \epsilon f(t)$ (and possibly some changes in other quantities). [In view of the limitations of the formalism of section IV F mentioned above, arguments other than t for the function f have been excluded.] The relation between the induced derivations Z_ξ on $\mathcal{A}$ and $\hat{Z}_\xi$ on $\mathcal{A}^e$ is given by

$$\hat{Z}_\xi = Z_\xi + f(t)\frac{\partial}{\partial t}. \tag{99}$$

We have $Z_\xi = Y_{h_\xi}$ (see section IV D). We look for the quantity $\hat{h}_\xi$ (the prospective Noether invariant) such that Eq.(85) holds. (Finding such a quantity will establish invariance of Ω under the relevant group action and also determine the corresponding Noether invariant.) Equations (99) and (81) now give the desired relation

$$\begin{aligned} i_{\hat{Z}_\xi}\Omega &= i_{Z_\xi}\tilde{\omega} - i_{Z_\xi}(dH)dt + f(t)dH \\ &= -dh_\xi - Y_{h_\xi}(H)dt + f(t)dH. \end{aligned} \tag{100}$$

Most of the equations in the following two subsections have the same mathematical form as some of the equations in Ref.[97,88a] and/or [85]. The following couple of remarks should serve to clarify the situation.
(a) Classical elementary systems are defined in terms of a transitive canonical action of the relevant relativity group on symplectic manifolds. These are obviously special cases of supmech elementary systems corresponding to commutative system algebras of the type treated in section

III H. Those results in the treatment of classical elementary systems whose derivation does not use the commutativity of the algebra $\mathcal{A}_{cl}$ are expected to be valid for general supmech elementary systems.
(b) Quantum elementary systems are defined in terms of projective unitary representations of the relevant relativity group on separable Hilbert spaces. Keeping the developments in section III J in view, these are seen as special cases of supmech elementary systems when the system algebra $\mathcal{A}$ is a member of a triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ [a 'quantum triple'; see section VII B] where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and $\mathcal{A}$ an algebra of operators having $\mathcal{D}$ as an invariant domain. According to theorem (3.2) in Ref.[88b], every projective unitary representation of a relativity group G_0 can be lifted to a unitary representation of the corresponding projective group $\hat{G}_0$ (called the 'projective covering group' of G_0 in that work). The infinitesimal generators of $\hat{G}_0$ arising in such a (continuous) unitary representation serve as hamiltonians of the corresponding supmech elementary system. Once the infinitesimal generators have been obtained, the Hilbert space goes into the background ; the rest of the work is algebraic. All the results obtained, in the traditional treatments of quantum elementary systems, by algebraic manipulations and use of the quantum Poisson brackets (54) are expected to be valid in the treatments of the corresponding elementary systems in supmech.

B. Nonrelativistic elementary systems

In this and the following subsection, we shall keep close to the notational conventions of Alonso [88a]. Our PBs, however, follow the conventions of Woodhouse [81] and differ from those of Ref.[85,88] by a sign; moreover, our Galilean generator **K** differs from that of Alonso by a sign.

The relativity group G_0 of the nonrelativistic domain of supmech is the Galilean group of transformations of the Newtonian space-time $R^3 \times R$ given by

$$x' = Rx + tv + a, \qquad t' = t + b \tag{101}$$

where $R \in SO(3)$, $v \in R^3$, $a \in R^3$ and $b \in R$. This group does not admit Poisson action. After a careful consideration of the freedom to modify the hamiltonians by additive terms, the hamiltonians J_i, K_i, P_i, H corresponding to the ten generators $\mathcal{J}_i, \mathcal{K}_i, \mathcal{P}_i (i = 1, 2, 3), \mathcal{H}$ of G_0 [so that $h_{\mathcal{P}_i} = P_i$ etc] satisfy the Poisson bracket relations [85]

$$\{J_i, J_j\} = -\epsilon_{ijk} J_k, \quad \{J_i, K_j\} = -\epsilon_{ijk} K_k, \quad \{J_i, P_j\} = -\epsilon_{ijk} P_k$$
$$\{K_i, H\} = -P_i, \quad \{K_i, P_j\} = -\delta_{ij} M, \tag{102}$$

where M is a neutral element; all other PBs vanish. The element $M \in \mathcal{A}$ has vanishing PBs with all the hamiltonian generators; this implies, by the argument presented above, M= mI, $m \in R$. We shall identify m as the mass of the elementary system. The condition $m \geq 0$ will follow later from an appropriate physical requirement. The objects P_i and J_i, being generators of the Euclidean subgroup E_3 of $\hat{G}_0$, are the momentum and angular momentum observables of section IV G.

Following the procedure outlined in section IV D, we augment the Lie algebra $\mathcal{G}_0$ of G_0 to a larger Lie algebra $\hat{\mathcal{G}}_0$ by including an additional generator $\mathcal{M}$ corresponding to M (which now becomes the Hamiltonian corresponding to $\mathcal{M} \in \hat{\mathcal{G}}_0$); it commutes with all other generators and appears only in the commutator

$$[\mathcal{K}_j, \mathcal{P}_k] = -\delta_{jk}\mathcal{M}. \tag{103}$$

The remaining commutation relations of $\hat{\mathcal{G}}_0$ are those of $\mathcal{G}_0$ ($[J_j, J_k] = -\epsilon_{jkl}J_l$ etc.). The projective group $\hat{G}_0$ of the Galilean group G_0 is the connected and simply connected Lie group with the Lie algebra $\hat{\mathcal{G}}_0$.

Representing a general group element of $\hat{G}_0$ in the form

$$\begin{aligned} g &= (A, v, b, a, \tau) \\ &= exp(-\tau\mathcal{M})exp(-a.\mathcal{P})exp(-b\mathcal{H})exp(-v.\mathcal{K})A \end{aligned} \tag{104}$$

where $A \in SU(2)$ and $\tau \in R$, the group law of $\hat{G}_0$ is obtained, after a straightforward calculation, as

$$g'g = (A'A, v' + R(A')v, b' + b, a' + bv' + R(A')a, \tau' + \tau + R(A')_{jk}v'_j a_k). \tag{105}$$

The transformation laws of the hamiltonians of $\hat{G}_0$ under its adjoint action may be found following the procedure of either [85] or [88a]. These transformation laws give the following three independent invariants :

$$M, \quad C_1 \equiv 2MH - \mathbf{P}^2, \quad C_2 \equiv (M\mathbf{J} - \mathbf{K} \times \mathbf{P})^2. \tag{106}$$

Of these, the first one is obvious; the vanishing of PBs of C_1 with all the hamiltonians is also easily checked. Writing $C_2 = B_j B_j$ where

$$B_j = MJ_j - \epsilon_{jkl}K_k P_l,$$

it is easily verified that

$$\{J_j, B_k\} = -\epsilon_{jkl}B_l, \quad \{K_j, B_k\} = \{P_j, B_k\} = \{H, B_k\} = 0$$

which finally leads to the vavishing of PBs of C_2 with all the hamiltonians. By the argument given above for M, the last two invariants also should be scalar multiples of the unit element in $\mathcal{A}$. The values of these three invariants characterize a Galilean elementary system in supmech.

We henceforth restrict ourselves to elementary systems with $m \neq 0$. Defining $X_i = m^{-1}K_i$, we have

$$\{X_j, X_k\} = 0, \quad \{P_j, X_k\} = \delta_{jk}I, \qquad \{J_j, X_k\} = -\epsilon_{jkl}X_l. \tag{107}$$

Comparing the last two equations above with the equations (94)(for n=3), we identify X_j with the position observables of section IV G. Note that the fact that the X_js mutually Poisson-commute comes from the relativity group.

Writing $\mathbf{S} = \mathbf{J} - \mathbf{X} \times \mathbf{P}$, we have $C_2 = m^2\mathbf{S}^2$. We have the PB relations

$$\{S_i, S_j\} = -\epsilon_{ijk}S_k, \quad \{S_i, X_j\} = 0 = \{S_i, P_j\}. \tag{108}$$

We identify $\mathbf{S}$ with the internal angular momentum or spin of the elementary system.

The invariant quantity

$$U \equiv \frac{C_1}{2m} = H - \frac{\mathbf{P}^2}{2m} \tag{109}$$

is interpreted as the *internal energy* of the elementary system. It is the appearance of this quantity (which plays no role in Newtonian mechanics) which is responsible for energy being defined in Newtonian mechanics only upto an additive constant.

Writing $\mathbf{S}^2 = s^2 I$ and U = u I, we see that Galilean elementary systems with $m \neq 0$ can be taken to be characterized/labelled by the parameters m, s and u. The fundamental kinematical observables are X_j, P_j and S_j (j=1,2,3). Other observables are assumed in supmech to be functions of the fundamental observables.

Henceforth we shall take u = 0 (a natural assumption to make if the elementary systems to be treated are particles). Eq.(109) now gives

$$H = \frac{\mathbf{P}^2}{2m} \tag{110}$$

which is the Hamiltonian for a free Galilean particle in supmech.

Note. (i) Full Galilean invariance (more generally, full invariance under a relativity group) applies only to an isolated system. Interactions/(external influences) are usually described with (explicit or implicit)reference to a fixed reference frame or a restricted class of frames. For example, the interaction described by a central potential implicitly assumes that the center of force is at the origin of axes of the chosen reference frame.
(ii) In the presence of external influences, translational invariance is lost and the PB $\{H, P_i\} = 0$ must be dropped. For a spinless particle, the Hamiltonian (assumed to be a function of the fundamental observables $\mathbf{X}$ and $\mathbf{P}$) then has the general form

$$H = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}, \mathbf{P}). \tag{111}$$

In most practical situations, V is a function of $\mathbf{X}$ only.

We can now rule out the case $m < 0$ on physical grounds because, by Eq.(110), this will allow arbitrarily large negative values for energy. (Expectation values of the observable $\mathbf{P}^2$ are expected to have no upper bound.)

Lastly, we consider the action of G_0 on the augmented algebra $\mathcal{A}^e$ for a free massive spinless particle. As noted above, it is adequate, for each ξ in the chosen basis of $\mathcal{G}_0$, to find a 'hamiltonian' $\hat{h}_\xi$ such that Eq.(85) holds; for this we must show the exactness of the form on the right hand side of Eq.(100). We have

(i)for rotations ($\xi = \mathcal{J}_i, h_\xi = J_i$) f(t) =0, $Y_\xi(H) = 0$, giving $\hat{h}_\xi = h_\xi = J_i$;

(ii) for Galilean boosts ($\xi = \mathcal{K}_i, h_\xi = K_i = mX_i$) f(t) = 0, $vY_\xi(H) = v\{K_i, H\} = -vP_i$ giving $\hat{h}_\xi = mX_i - P_i t$;

(iii)for space translations ($\xi = \mathcal{P}_i, h_\xi = P_i$) f(t) =0, $Y_\xi(H) = 0$, giving $\hat{h}_\xi = h_\xi = P_i$;

(iv) for time translations ($\xi = \mathcal{H}, h_\xi = H$) f(t,.) =1, $Y_\xi = 0$, giving $\hat{h}_\xi = H$.

Finally, the Noether invariants of the Galilean group are

$$\mathbf{J}, \quad m\mathbf{X} - \mathbf{P}t, \quad \mathbf{P}, \quad H \tag{112}$$

which are (up-to signs) the supmech avatars of those in Ref.(89).

C. Relativistic elementary systems

In the scheme of special relativity, the relativity group G_0 is the (identity component of) Poincaré group of transformations on the Minkowski space-time [$(R^4, \eta_{\mu\nu})$ where μ, ν =0,1,2,3 and $\eta_{\mu\nu}$ = diag(-1,1,1,1)]:

$$x' = \Lambda x + a, \quad \Lambda \in SO(3,1) \ \ (\text{with}\Lambda^0{}_0 \geq 1), \quad a \in R^4. \tag{113}$$

This group admits Poisson actions and the effective relativity group $\hat{G}_0$ is just the universal covering group of G_0 which has the group law

$$(A_2, a_2)(A_1, a_1) = (A_2 A_1, \Lambda(A_2)a_1 + a_2) \tag{114}$$

where $\Lambda(A)$ denotes the Lorentz group element corresponding to $A \in SL(2, C)$.

As the general method (of treating elementary systems in supmech) has been illustrated in the previous subsection, we shall be relatively more brief here. Some more details relevant to the algebraic work involved here may be found in Ref[97,88a,85]. We shall generally keep close to the developments in Ref[97,88a].

The generators of $\hat{G}_0$ are $\{\mathcal{M}_{\mu\nu} = -\mathcal{M}_{\nu\mu}, \mathcal{P}_\mu\}$ or, equivalently, $\{\mathcal{J}_i, \mathcal{K}_i, \mathcal{P}_i, \mathcal{H}\}$ where $\mathcal{P}^0 = \mathcal{H}, \mathcal{M}_{0i} = \mathcal{K}_i$ and $\mathcal{M}_{ij} = \epsilon_{ijk}\mathcal{J}_k$. The hamiltonians (J_i, K_i, P_i, H) arising from the Poisson action of $\hat{G}_0$ on the symplectic superalgebra $(\mathcal{A}, \omega)$ of an elementary system satisfy the PB relations

$$\begin{aligned} \{J_i, J_j\} = -\epsilon_{ijk}J_k, \quad \{J_i, K_j, \} = -\epsilon_{ijk}K_k, \quad \{J_i, P_j\} = -\epsilon_{ijk}P_k, \\ \{K_i, K_j\} = \epsilon_{ijk}J_k, \quad \{K_i, P_j\} = -\delta_{ij}H, \quad \{K_i, H\} = -P_i; \end{aligned} \tag{115}$$

all other PBs vanish. The PBs for the manifestly Lorentz-covariant hamiltonians $M_{\mu\nu}, P_\mu$ are those of Eq.(92) with j,k,l,p,q replaced by $\mu, \nu, \lambda, \rho, \sigma$ and δ_{jl} by $\eta_{\mu\lambda}$ etc.

Defining $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}M_{\nu\lambda}P_\rho$ (the Pauli-Lubanski vector), the two independent invariants of the $\hat{G}_0$-action are $P^2 \equiv P^\mu P_\mu$ and W^2; this can be directly checked from the covariant PBs mentioned above. For elementary systems, they take values $p^2 I$ and $w^2 I$ (where I is the unit

element of the algebra $\mathcal{A}$); the real-valued quantities p^2 and w^2 characterize the elementary systems. It is useful to note that

$$W^0 = -\mathbf{J}.\mathbf{P}, \quad \mathbf{W} = -H\mathbf{J} + \mathbf{P} \times \mathbf{K}. \tag{116}$$

We shall restrict ourselves to the cases with $p^2 \leq 0$ (i.e. p^μ non-spacelike) and write $p^2 = -m^2$ (with $m \geq 0$). For situations with $m > 0$ and H and (H + mI) invertible, one can define the position and spin observables as follows :

$$\mathbf{X} = -\frac{1}{2}[\mathbf{K}, H^{-1}]_+ + [mH(H+mI)]^{-1}\mathbf{P} \times \mathbf{W} \tag{117}$$

$$\mathbf{S} = -m^{-1}\mathbf{W} + [mH(H+mI)]^{-1}\mathbf{W}.\mathbf{PP}. \tag{118}$$

The expected PB relations hold :

$$\{X_i, X_j\} = 0, \quad \{P_i, X_j\} = \delta_{ij} I,$$
$$\{X_i, S_j\} = 0 = \{P_i, S_j\}$$
$$\{J_i, X_j\} = -\epsilon_{ijk} X_k, \quad \{J_i, S_j\} = \{S_i, S_j\} = -\epsilon_{ijk} S_k. \tag{119}$$

We have $W^2 = m^2\mathbf{S}^2$ and the relations

$$H^2 = \mathbf{P}^2 + m^2 I, \quad \mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S}$$
$$\mathbf{K} = -\frac{1}{2}[\mathbf{X}, H]_+ + (H+mI)^{-1}\mathbf{S} \times \mathbf{P}. \tag{120}$$

Writing $|\mathbf{S}|^2 = s^2 I$ (with $s \geq 0$), the invariant quantities quantities m (mass) and s (spin) characterize an elementary system and the fundamental kinematical observables are again X_j, P_j, S_j(j= 1,2,3).

VI. COUPLED SYSTEMS IN SUPMECH

We shall now consider the interaction of two systems S_1 and S_2 described individually as the supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ (i=1,2) and treat the coupled system $S_1 + S_2$ also as a supmech Hamiltonian system. To facilitate this, we must obtain the relevant mathematical objects for the coupled system.

A. The symplectic form and Poisson bracket on the tensor product of two superalgebras

The superalgebra corresponding to the coupled system ($S_1 + S_2$) will be taken as the (skew) tensor product $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$; its elements are finite sums of tensored pairs :

$$\sum_{j=1}^{m} A_j \otimes B_j \qquad A_j \in \mathcal{A}^{(1)}, \quad B_j \in \mathcal{A}^{(2)}$$

with the multiplication rule

$$(\sum_{j=1}^{m} A_j \otimes B_j)(\sum_{k=1}^{n} A_k \otimes B_k) = \sum_{j,k} \eta_{B_j A_k}(A_j A_k) \otimes (B_j B_k). \tag{121}$$

The superalgebra $\mathcal{A}^{(1)}$ (resp. $\mathcal{A}^{(2)}$) has, in $\mathcal{A}$, an isomorphic copy consisting of the elements $(A\otimes I_2, A\in\mathcal{A}^{(1)})$ (resp. $I_1\otimes B, B\in\mathcal{A}^{(2)}$) to be denoted as $\tilde{\mathcal{A}}^{(1)}$ (resp. $\tilde{\mathcal{A}}^{(2)}$). Here I_1 and I_2 are the unit elements of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively. We shall also use the notations $\tilde{A}^{(1)}=A\otimes I_2$ and $\tilde{B}^{(2)}=I_1\otimes B$.

Derivations and differential forms on $\mathcal{A}^{(i)}$ and $\tilde{\mathcal{A}}^{(i)}$ (i = 1,2) are formally related through the induced mappings corresponding to the isomorphisms $\Xi^{(i)}:\mathcal{A}^{(i)}\to\tilde{\mathcal{A}}^{(i)}$ given by $\Xi^{(1)}(A)=A\otimes I_2$ and $\Xi^{(2)}(B)=I_1\otimes B$. For example, corresponding to $X\in SDer(\mathcal{A}^{(1)})$, we have $\tilde{X}^{(1)}=\Xi^{(1)}_*(X)$ in SDer$(\tilde{\mathcal{A}}^{(1)})$ given by [see Eq.(3)]

$$\tilde{X}^{(1)}(\tilde{A}^{(1)})=\Xi^{(1)}_*(X)(\tilde{A}^{(1)})=\Xi^{(1)}[X(A)]=X(A)\otimes I_2. \tag{122}$$

Similarly, corresponding to $Y\in SDer(\mathcal{A}^{(2)})$, we have $\tilde{Y}^{(2)}\in$ SDer$(\tilde{\mathcal{A}}^{(2)})$ given by $\tilde{Y}^{(2)}(\tilde{B}^2)=I_1\otimes Y(B)$. For the 1-forms $\alpha\in\Omega^1(\mathcal{A}^{(1)})$ and $\beta\in\Omega^1(\mathcal{A}^{(2)})$, we have $\tilde{\alpha}^{(1)}\in\Omega^1(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{\beta}^{(2)}\in\Omega^1(\tilde{\mathcal{A}}^{(2)})$ given by [see Eq.(26)]

$$\tilde{\alpha}^{(1)}(\tilde{X}^{(1)})=\Xi^{(1)}[\alpha(((\Xi^{(1)})^{-1})_*\tilde{X}^{(1)})]=\Xi^{(1)}[\alpha(X)]=\alpha(X)\otimes I_2 \tag{123}$$

and $\tilde{\beta}^{(2)}(\tilde{Y}^{(2)})=I_1\otimes\beta(Y)$. Analogous formulas hold for the higher forms.

We can extend the action of the superderivations $\tilde{X}^{(1)}\in SDer(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{Y}^{(2)}\in SDer(\tilde{\mathcal{A}}^{(2)})$ to $\tilde{\mathcal{A}}^{(2)}$ and $\tilde{\mathcal{A}}^{(1)}$ respectively by defining

$$\tilde{X}^{(1)}(\tilde{B}^{(2)})=0,\quad \tilde{Y}^{(2)}(\tilde{A}^{(1)})=0\ \text{ for all }\ A\in\mathcal{A}^{(1)}\text{ and }\ B\in\mathcal{A}^{(2)}. \tag{124}$$

Note that an $X\in SDer(\mathcal{A})$ is determined completely by its action on the subalgebras $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$:

$$X(A\otimes B)=X(\tilde{A}^{(1)}\tilde{B}^{(2)})=(X\tilde{A}^{(1)})\tilde{B}^{(2)}+\eta_{XA}\tilde{A}^{(1)}X(\tilde{B}^{(2)}).$$

With the extensions described above, we have available to us superderivations belonging to the span of terms of the form [see Eq.(122)]

$$X=X^{(1)}\otimes I_2+I_1\otimes X^{(2)}. \tag{125}$$

Replacing I_2 and I_1 in Eq.(125) by elements of $Z(\mathcal{A}^{(2)})$ and $Z(\mathcal{A}^{(1)})$ respectively, we again obtain superderivations of $\mathcal{A}$. We, therefore, have the space of superderivations

$$[SDer(\mathcal{A}^{(1)})\otimes Z(\mathcal{A}^{(2)})]\oplus[Z(\mathcal{A}^{(1)})\otimes SDer(\mathcal{A}^{(2)})]. \tag{126}$$

This space, however, is generally only a Lie sub-superalgebra of SDer$(\mathcal{A})$. For example, for $\mathcal{A}^{(1)}=M_m(C)$ and $\mathcal{A}^{(2)}=M_n(C)$, recalling that all the derivations of these matrix algebras are inner and that their centers consist of scalar multiples of the respective unit matrices , we have the (complex) dimensions of SDer$(\mathcal{A}^{(1)})$, and SDer$(\mathcal{A}^{(2)})$ respectively, (m^2-1) and (n^2-1) [so that the dimension of the space (126) is m^2+n^2-2] whereas that of $SDer(\mathcal{A})$ is (m^2n^2-1).

It is instructive to obtain explicit representation(s) for a general derivation of the matrix algebra $\mathcal{A} = M_m(C) \otimes M_n(C)$. We have

$$[A \otimes B, C \otimes D]_{ir,js} = A_{ik}B_{rt}C_{kj}D_{ts} - C_{ik}D_{rt}A_{kj}B_{ts}$$

which gives

$$\begin{aligned} [A \otimes B, C \otimes D]_- &= AC \otimes BD - CA \otimes DB & (127) \\ &= [A, C]_- \otimes \frac{1}{2}[B, D]_+ + \frac{1}{2}[A, C]_+ \otimes [B, D]_-. & (128) \end{aligned}$$

This gives, in obvious notation,

$$\begin{aligned} D_{A\otimes B} \equiv [A \otimes B, .]_- &= A.(.) \otimes B_{.}(.) - (.).A \otimes (.).B & (129) \\ &= D_A \otimes J_B + J_A \otimes D_B & (130) \end{aligned}$$

where J_B is the linear mapping on $\mathcal{A}^{(2)}$ given by $J_B(D) = \frac{1}{2}[B, D]_+$ and a similar expression for J_A as a linear mapping on $\mathcal{A}^{(1)}$. Eq.(129) shows that a derivation of the algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ need not explicitly contain those of $\mathcal{A}^{(i)}$. We shall, however, not get involved in the search for the most general expression for a derivation of the tensor product algebra $\mathcal{A}$(although such an expression would be very useful). The expression (130) is more useful for us; it is a special case of the more general form

$$X = X_1 \otimes \Psi_2 + \Psi_1 \otimes X_2 \tag{131}$$

where $X_i \in SDer(\mathcal{A}^{(i)})$ (i=1,2) and $\Psi_i : \mathcal{A}^{(i)} \to \mathcal{A}^{(i)}$ (i =1,2) are linear mappings. Our constructions below will lead us to the form (131). It is important to note, however, that an expression of the form (131) (which represents a linear mapping of $\mathcal{A}$ into itself) need not always be a derivation as can be easily checked. We shall impose the condition (1) on such an expression to obtain a derivation.

To obtain the differential forms and the exterior product on $\mathcal{A}$, the most straightforward procedure is to obtain the graded differential space $(\Omega(\mathcal{A}), d)$ as the tensor product [98] of the graded differential spaces $(\Omega(\mathcal{A}^{(1)}), d_1)$ and $(\Omega(\mathcal{A}^{(2)}), d_2)$. A (homogeneous) differential k-form on $\mathcal{A}$ is of the form (in obvious notation)

$$\alpha_{kt} = \sum_{\substack{i+j=k \\ r+s=t \ mod(2)}} \alpha^{(1)}_{ir} \otimes \alpha^{(2)}_{js}. \tag{132}$$

The d operation on $\Omega(\mathcal{A})$ is given by [here $\alpha \in \Omega^p(\mathcal{A}^{(1)})$ and $\beta \in \Omega(\mathcal{A}^{(2)})$]

$$d(\alpha \otimes \beta) = (d_1\alpha) \otimes \beta + (-1)^p \alpha \otimes d_2\beta. \tag{133}$$

Given the symplectic forms $\omega^{(i)}$ on $\mathcal{A}^{(i)}$ (i=1,2) we shall construct the induced symplectic form ω on $\mathcal{A}$satisfying the following conditions :

(a) It should not depend on anything other than the objects $\omega^{(i)}$ and $I_{(i)}$ (i=1,2) [the 'naturality'/'canonicality' assumption for ω (the unit elements are the only distinguished elements of the algebras being considered)].
(b) The restrictions of ω to $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$ be, respectively, $\omega^{(1)} \otimes I_2$ and $I_1 \otimes \omega^{(2)}$.
this determines ω uniquely :

$$\omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}. \tag{134}$$

To verify that it is a symlectic form, we must show that it is (i) closed and (ii) nondegenerate. Eq.(133) gives

$$d\omega = (d_1\omega^{(1)}) \otimes I_2 + \omega^{(1)} \otimes d_2(I_2) + d_1(I_1) \otimes \omega^{(2)} + I_1 \otimes d_2\omega^{(2)} = 0$$

showing that ω is closed. To show the nondegeneracy of ω, we must show that, given $A \otimes B \in \mathcal{A}$, there exists a unique superderivation $Y = Y_{A\otimes B}$ in $SDer(\mathcal{A})$ such that

$$\begin{aligned} i_Y\omega = -d(A \otimes B) &= -(d_1A) \otimes B - A \otimes d_2B \\ &= i_{Y_A^{(1)}}\omega^{(1)} \otimes B + A \otimes i_{Y_B^{(2)}}\omega^{(2)}. \end{aligned} \tag{135}$$

where $Y_A^{(1)}$ and $Y_B^{(2)}$ are the Hamiltonian superderivations associated with $A \in \mathcal{A}^{(1)}$ and $B \in \mathcal{A}^{(2)}$. The structure of Eq.(135) suggests that Y must be of the form [see Eq.(131)]

$$Y = Y_A^{(1)} \otimes \Psi_B^{(2)} + \Psi_A^{(1)} \otimes Y_B^{(2)} \tag{136}$$

where the linear mappings $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ satisfy the conditions $\Psi_A^{(1)}(I_1) = A$ and $\Psi_B^{(2)}(I_2) = B$. Recalling the discussion after Eq.(131) and Eq.(1) [and denoting the multiplication operators in $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ and $\mathcal{A}$ by μ_1, μ_2 and μ respectively], the condition for Y to be a derivation may be written as

$$Y \circ \mu(C \otimes D) - \eta_{Y,C\otimes D}\mu(C \otimes D) \circ Y = \mu(Y(C \otimes D)). \tag{137}$$

Noting that $\mu(C \otimes D) = \mu_1(C) \otimes \mu_2(D)$, Eq.(137) with Y of Eq.(136) gives

$$\begin{aligned} &\eta_{BC}\{[Y_A^{(1)} \circ \mu_1(C)] \otimes [\Psi_B^{(2)} \circ \mu_2(D)] + [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \circ \mu_2(D)]\} \\ -(-1)^{\epsilon}&\{[\mu_1(C) \circ Y_A^{(1)}] \otimes [\mu_2(D) \circ \Psi_B^{(2)}] + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes [\mu_2(D) \circ Y_B^{(2)}]\} \\ &= \eta_{BC}[\mu_1(\{A, C\}_1) \otimes \mu_2(\Psi_B^{(2)}(D)) + \mu_1(\Psi_A^{(1)}(C)) \otimes \mu_2(\{B, D\}_2)] \end{aligned} \tag{138}$$

where $\epsilon \equiv \epsilon_A\epsilon_C + \epsilon_B\epsilon_D + \epsilon_B\epsilon_C$ and we have used the relations $Y_A^{(1)}(C) = \{A, C\}_1$ and $Y_B^{(2)}(D) = \{B, D\}_2$.

The objects $Y_A^{(1)}$ and $Y_B^{(2)}$, being superderivations, satisfy relations of the form (1) :

$$\begin{aligned} Y_A^{(1)} \circ \mu_1(C) - \eta_{AC}\mu_1(C) \circ Y_A^{(1)} = \mu_1(Y_A^{(1)}(C)) = \mu_1(\{A, C\}_1) \\ Y_B^{(2)} \circ \mu_2(D) - \eta_{BD}\mu_2(D) \circ Y_B^{(2)} = \mu_2(\{B, D\}_2). \end{aligned} \tag{139}$$

Putting $D = I_2$ in Eq.(138), we have [noting that $\mu_2(D) = \mu_2(I_2) = id_2$, the identity mapping on $\mathcal{A}^{(2)}$ and that $\{B, I_2\}_2 = Y_B^{(2)}(I_2) = 0$]

$$\begin{aligned}[Y_A^{(1)} \circ \mu_1(C)] \otimes \Psi_B^{(2)} \quad + \quad & [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \\ & -\eta_{AC}\{[\mu_1(C) \circ Y_A^{(1)}] \otimes \Psi_B^{(2)} + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}\} \\ & = \mu_1(\{A, C\}_1) \otimes \mu_2(B) \end{aligned} \tag{140}$$

which, along with equations (139), gives

$$\begin{aligned}\mu_1(\{A, C\}_1) \otimes [\Psi_B^{(2)} - \mu_2(B)] \quad = \quad & \\ -[\Psi_A^{(1)} \circ \mu_1(C) \quad - \quad & \eta_{AC}\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}. \end{aligned} \tag{141}$$

Similarly, putting $C = I_1$ in Eq.(138), we get

$$\begin{aligned}[\Psi_A^{(1)} - \mu_1(A)] \otimes \mu_2(\{B, D\}_2) \quad = \quad & \\ -Y_A^{(1)} \otimes [\Psi_B^{(2)} \circ \mu_2(D) \quad - \quad & \eta_{BD}\mu_2(D) \circ \Psi_B^{(2)}]. \end{aligned} \tag{142}$$

Now, equations (142) and (141) give the relations

$$\Psi_A^{(1)} - \mu_1(A) = \lambda_1 Y_A^{(1)} \tag{143}$$

$$\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD}\mu_2(D) \circ \Psi_B^{(2)} = -\lambda_1 \mu_2(\{B, D\}_2) \tag{144}$$

$$\Psi_B^{(2)} - \mu_2(B) = \lambda_2 Y_B^{(2)} \tag{145}$$

$$\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC}\mu_1(C) \circ \Psi_A^{(1)} = -\lambda_2 \mu_1(\{A, C\}_1) \tag{146}$$

where λ_1 and λ_2 are complex numbers.

Equations (136), (143) and (145) now give

$$\begin{aligned} Y \quad &= \quad Y_A^{(1)} \otimes [\mu_2(B) + \lambda_2 Y_B^{(2)}] + [\mu_1(A) + \lambda_1 Y_A^{(1)}] \otimes Y_B^{(2)} \\ &= \quad Y_A^{(1)} \otimes \mu_2(B) + \mu_1(A) \otimes Y_B^{(2)} + (\lambda_1 + \lambda_2) Y_A^{(1)} \otimes Y_B^{(2)}. \end{aligned} \tag{147}$$

Note that only the combination $(\lambda_1 + \lambda_2) \equiv \lambda$ appears in Eq.(147). To have a unique Y, we must obtain an equation fixing λ in terms of given quantities.

Substituting for $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ from equations (143) and (145) into equations (144) and (146) and using equations (139), we obtain the equations

$$\lambda \mu_1(\{A, C\}_1) \quad = \quad -\mu_1([A, C]) \quad \text{for all} \quad A, C \in \mathcal{A}^{(1)} \tag{148}$$

$$\lambda \mu_2(\{B, D\}_2) \quad = \quad -\mu_2([B, D]) \quad \text{for all} \quad B, D \in \mathcal{A}^{(2)}. \tag{149}$$

We have not one but two equations of the type we have been looking for. This is a signal for the emergence of nontrivial conditions (for the desired symplectic structure on the tensor product superalgebra to exist).

Let us consider the equations (148,149) for the various possible situations (corresponding to whether or not one or both the superalgebras are super-commutative) :

(i) Let $\mathcal{A}^{(1)}$ be supercommutative. Assuming that the PB $\{,\}_1$ is nontrivial, Eq.(148) implies that $\lambda = 0$. Eq.(149) then implies that $\mathcal{A}^{(2)}$ must also be super-commutative. It follows that
(a) when both the superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are super-commutative, the unique Y is given by Eq.(147) with $\lambda = 0$;
(b) a 'natural'/'canonical' symplectic structure does not exist on the tensor product of a super-commutative and a non-supercommutative superalgebra.

(ii) Let the superalgebra $\mathcal{A}^{(1)}$ be non-supercommutative. Eq.(148) then implies that $\lambda \neq 0$, which, along with Eq.(149) implies that the superalgebra $\mathcal{A}^{(2)}$ is also non-supercommutative [which is also expected from (b) above]. Equations (148,149) now give

$$\{A, C\}_1 = -\lambda^{-1}[A, C], \quad \{B, D\}_2 = -\lambda^{-1}[B, D] \tag{150}$$

which shows that, when both the superalgebras are non-supercommutative, a 'natural'/'canonical' symplectic structure on their (skew) tensor product exists if and only if each superalgebra has a quantum symplectic structure with the *same* parameter $(-\lambda)$, i.e.

$$\omega^{(1)} = -\lambda\omega_c^{(1)}, \quad \omega^{(2)} = -\lambda\omega_c^{(2)} \tag{151}$$

where $\omega_c^{(i)}$ (i=1,2) are the canonical symplectic forms on the two superalgebras. The traditional quantum symplectic structure is obtained with $\lambda = i\hbar$.

Note. The two forms $\omega^{(i)}$(i=1,2) of Eq.(151) represent bonafide symplectic structures only if the superalgebras $\mathcal{A}^{(i)}$ (i=1,2) have only inner superderivations (see section III I). More generally, we can have generalized symplectic superalgebras $(\mathcal{A}^{(i)}, \mathcal{X}^{(i)}, \omega^{(i)})$ (i=1,2) where $\mathcal{X}^{(i)} = ISDer(\mathcal{A}^{(i)})$.

In all the permitted cases, the PB on the superalgebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ is given by

$$\begin{aligned} \{A \otimes B, C \otimes D\} = Y_{A\otimes B}(C \otimes D) \quad &= \quad \eta_{BC}[\{A, C\}_1 \otimes BD + AC \otimes \{B, D\}_2 \\ &\quad +\lambda\{A, C\}_1 \otimes \{B, D\}_2] \end{aligned} \tag{152}$$

where the parameter λ vanishes in the super-commutative case; in the non-supercommutative case, it is the universal parameter appearing in the symplectic forms (151).

Noting that, in the non-supercommutative case,

$$\begin{aligned} \lambda\{A, C\}_1 \otimes \{B, D\}_2 &= -[A, C] \otimes \{B, D\}_2 = -\{A, C\}_1 \otimes [B, D] \\ &= -\frac{1}{2}[A, C] \otimes \{B, D\}_2 - \frac{1}{2}\{A, C\}_1 \otimes [B, D], \end{aligned} \tag{153}$$

the PB of Eq.(152) can be written in the more symmetric form

$$\begin{aligned} &\{A \otimes B, C \otimes D\} \\ &= \eta_{BC}[\{A, C\}_1 \otimes \tfrac{BD+\eta_{BD}DB}{2} + \tfrac{AC+\eta_{AC}CA}{2} \times \{B, D\}_2]. \end{aligned} \tag{154}$$

Recalling that, for the matrix algebra $M_n(C)(n \geq 2)$, the Poisson bracket (with the canonical symplectic form) is a commutator, Eq.(128) is a special case of Eq.(154). In fact, had we employed supermatrices, we would have got exactly Eq.(154) as can be easily verified using the equation preceding Eq.(127). As shown below, a direct calculation for the tensor product of two classical algebras of observables also gives results consistent with Eq.(154).

Example [Both algebras commutative]

$$\mathcal{A}^{(1)} = C^\infty(R^m), \quad \mathcal{A}^{(2)} = C^\infty(R^n); \qquad \text{(m,n even)}.$$

Let x^i and y^r be the coordinates on R^m and R^n respectively and let the Poisson brackets on them be

$$\{f,g\}_1 = \omega_1^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}; \quad \{u,v\}_2 = \omega_2^{rs}\frac{\partial u}{\partial y^r}\frac{\partial v}{\partial y^s}.$$

Let $z^a = (x^i, y^r)$ be the coordinates on $R^m \times R^n = R^{m+n}$. The PB on $\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} = C^\infty(R^{m+n})$ is [putting $F(z) \equiv (f \otimes u)(x,y) = f(x)u(y)$ and G(z) = g(x)v(y) and choosing the symplectic form on R^{m+n} in accordance with Eq.(134)]

$$\begin{aligned} \{F,G\} &= \omega^{ab}\frac{\partial F}{\partial z^a}\frac{\partial G}{\partial z^b} \\ &= \omega_1^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}uv + \omega_2^{rs}\frac{\partial u}{\partial y^r}\frac{\partial v}{\partial y^s}fg \\ &= \{f,g\}_1 uv + \{u,v\}_2 fg, \end{aligned} \tag{155}$$

which is consistent with Eq.(154).

B. Dynamics of coupled systems

Given the individual systems S_1 and S_2 as the supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ (i = 1,2), the coupled system $(S_1 + S_2)$ is a supmech Hamiltonian system with the system algebra and symplectic form as discussed above and the Hamiltonian H given by

$$H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int} \tag{156}$$

where the interaction Hamiltonian is generally of the form

$$H_{int} = \sum_{i=1}^{n} F_i \otimes G_i. \tag{157}$$

The evolution (in the Heisenberg type picture) of a typical obsevable $A(t) \otimes B(t)$ is governed by the supmech Hamilton's equation

$$\begin{aligned} \frac{d}{dt}[A(t) \otimes B(t)] &= \{H, A(t) \otimes B(t)\} \\ &= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2 \\ &\quad + \{H_{int}, A(t) \otimes B(t)\}. \end{aligned} \tag{158}$$

The last Poisson bracket in this equation can be evaluated using Eq.(152) or (154). When both the systems are quantum systems, the equations above reduce to those of traditional QM in the Heisenberg picture and when both are classical, they reduce to those of the classical Hamiltonian formalism.

(In the Schrödinger type picture) the time evolution of states of the coupled system is given by the supmech Liouville equation (65) with the Hamiltonian of Eq.(156).

In favorable situations, the supmech Heisenberg or Liouville equations may be written for finite time intervals by using appropriate exponentiations of operators. We shall be doing this in section VIII below in which a concrete application of the formalism of this section to measurements in quantum mechanics will be described.

The main lesson from this section is that *all* systems in nature whose interaction with other systems can be talked about must belong to one of the two 'worlds' : the 'commutative world' in which all system superalgebras must be super-commutative and the 'noncommutative world' in which all system superalgebras must be non-supercommutative with a *universal* quantum symplectic structure. (There is no restriction on the type of symplectic structure on system superalgebras in the commutative world.) In view of the familiar inadequacy of the commutative world, the 'real' world must clearly be the noncommutative (hence quantum) world; its systems will be called quantum systems. (This is formalized as axiom **A7** in section IX.) The classical systems with commutative system algebras and traditional symplectic structures will appear only in the appropriately defined classical limit (or, more generally, in the classical approximation) of quantum systems.

This brings us on the threshold of an autonomous development of QM.

VII. QUANTUM SYSTEMS, (SUPER-)CLASSICAL SYSTEMS AND QUANTUM-CLASSICAL CORRESPONDENCE

We start by describing what we call 'standard quantum systems' (eventually to be seen as quantum systems without superselection rules) in purely algebraic terms.

A. Standard quantum systems

By a *standard quantum system* (SQS) we shall mean a supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1, \omega, H)$ in which the system algebra $\mathcal{A}$ is special (in the sense of section III I) and has a trivial graded center and ω is the *quantum symplectic form* ω_Q given by [see Eq.(53)]

$$\omega_Q = -i\hbar\omega_c. \tag{159}$$

(We have, in the terminology of section III I, the quantum symplectic structure with parameter $b = -i\hbar$.) This is the only place where we put the Planck constant 'by hand' (obviously the most natural place to do it); its appearance at all conventional places (canonical commutation relations, Heisenberg and Schrödinger equations, etc) will be automatic.

The *quantum Poisson bracket* implied by the quantum symplectic form is [see Eq.(54)]

$$\{A, B\} = (-i\hbar)^{-1}[A, B]. \tag{160}$$

Recall that the bracket [,] represents a supercommutator; it follows that the bracket on the right in Eq.(160) is an anticommutator when both A and B are odd/fermionic and a commutator otherwise.

A *quantum canonical transformation* is an automorphism Φ of the system algebra $\mathcal{A}$ such that $\Phi^*\omega_Q = \omega_Q$. Now

$$(\Phi^*\omega_Q)(X_1, X_2) = \Phi^{-1}[\omega_Q(\Phi_* X_1, \Phi_* X_2)] \tag{161}$$

where X_1, X_2 are inner superderivations, say, D_A and D_B. We have

$$(\Phi_* D_A)(B) = \Phi[D_A(\Phi^{-1}(B)] = \Phi([A, \Phi^{-1}(B)]) = [\Phi(A), B]$$

which gives

$$\Phi_* D_A = D_{\Phi(A)}. \tag{162}$$

Eq.(161) now gives

$$\Phi(i[A, B]) = i[\Phi(A), \Phi(B)] \tag{163}$$

which shows, quite plausibly, that quantum canonical transformations are (in the present algebraic setting — we have not yet come to the Hilbert space) the automorphisms of the system algebra preserving the quantum PBs.

The evolution of an SQS in time is governed, in the Heisenberg picture, by the supmech Hamilton's equation (64) which now becomes the familiar *Heisenberg equation* of motion

$$\frac{dA(t)}{dt} = (-i\hbar)^{-1}[H, A(t)]. \tag{164}$$

In the Schrödinger picture, the time dependence is carried by the states and the evolution equation (65) takes the form

$$\frac{d\phi(t)}{dt}(A) = (-i\hbar)^{-1}\phi(t)([H, A]) \tag{165}$$

which may be called the *generalized von Neumann equation.*

We shall call two SQSs $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$ and $\Sigma' = (\mathcal{A}', \mathcal{S}_1', \omega', H')$ equivalent if they are equivalent as supmech Hamiltonian systems. (See section IV C.) The main requirement is the isomorphism of $\mathcal{A}$ and $\mathcal{A}'$ as (special) topological algebras; the rest is almost automatic.
Note. In the abstract algebraic framework, the CC condition is to be kept track of. An advantage, as we shall see below, of the Hilbert space based realization of quantum systems is that the CC condition is automatically satisfied.

B. Hilbert space based realizations of standard quantum systems

Recalling the brief treatment of the Schrödinger representation in section III J, it is useful to introduce the concept of a *quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H}$ is a complex Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and $\mathcal{A}$ a *-algebra of operators having $\mathcal{D}$ as an invariant domain [which means that $\mathcal{D}$ is contained in the domain of each element A of $\mathcal{A}$ (and of its adjoint $A^\dagger$) and is mapped into itself by every such A and $A^\dagger$]. The *-operation on $\mathcal{A}$ is defined to be the restriction of the adjoint operation to $\mathcal{D}$. We shall assume that, for a given $\mathcal{A}$, $\mathcal{D}$ is maximal, i.e. largest such domain. When $\mathcal{A}$ is generated by a finite set of fundamental observables $F_1, .., F_n$, then in the notation of Ref.[51], $\mathcal{D} = C^\infty(F_1, .., F_n)$ (i.e. intersection of the domains of all polynomials in $F_1, .., F_n$).

If, in the quantum triple above, we take $\mathcal{A}$ as our system algebra, then its states are given by the subclass of density operators ρ on $\mathcal{H}$ for which $|Tr(\rho A)| < \infty$ for all observables A in $\mathcal{A}$; pure states are the subclass of these consisting of one-dimensional projection operators. In view of the maximality of $\mathcal{D}$, the latter are precisely the one-dimensional projectors $|\psi >< \psi|$ where ψ is any normalized element of $\mathcal{D}$ (which means that pure states are the unit rays corresponding to the elements of $\mathcal{D}$).

When the algebra $\mathcal{A}$ of the quantum triple above is special, we obtain a Hilbert space based SQS by choosing the quantum symplectic form as above and an even Hermitian element H of $\mathcal{A}$ as the Hamiltonian. It is clear that, when the choice of Hamiltonian is not under consideration, a Hilbert space-based SQS is adequately described as a quantum triple with the algebra $\mathcal{A}$ qualified as above.

The CC condition for the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1)$ can be explicitly verified for the Hilbert space based SQSs. :
(i) Given $A, B \in \mathcal{O}(\mathcal{A})$, and $(\psi, A\psi) = (\psi, B\psi)$ for all normalized ψ in $\mathcal{D}$ (hence for all ψ in $\mathcal{D}$), we have $(\phi, A\psi) = (\phi, B\psi)$ for all $\phi, \psi \in \mathcal{D}$, implying A = B. [Hint : Consider the given equality with the states $(\phi + \psi)/\sqrt{2}$ and $(\phi + i\psi)/\sqrt{2}$.]
(ii) Given normalized vectors ψ_1, ψ_2in $\mathcal{D}$ and $(\psi_1, A\psi_1) = (\psi_2, A\psi_2)$ for all $A \in \mathcal{O}(\mathcal{A})$, the equality $\psi_1 = \psi_2$ (upto a phase) can be seen by taking, for A, the projection operators corresponding to members of an orthonormal basis in $\mathcal{H}$ containing ψ_1 as a member.

Note. We have implicitly assumed above that all elements of $\mathcal{D}$ represent pure states. This excludes the situations when $\mathcal{H}$ is a direct sum of more than one coherent subspaces in the presence of superselection rules.

An interesting feature of the Hilbert space-based SQSs is that we have density operators representing states which, being Hermitian operators, are also observables. A density operator ρ_ϕ representing a state ϕ is the observable corresponding to the property of being in the state ϕ. Given two states represented by density operators ρ_1 and ρ_2, we have the quantity $w_{12} = Tr(\rho_1\rho_2)$ defined (representing the expectation value of the observable ρ_1 in the state ρ_2 and and vice versa) which has the natural interpretation of transition probability from one

of the states to the other (the two are equal because $w_{12} = w_{21}$). When $\rho_i = |\psi_i >< \psi_i|$ (i = 1,2) are pure states, we have $Tr(\rho_1\rho_2) = | < \psi_1|\psi_2 > |^2$ — the familiar text book expression for the transition probability between two pure quantum states.

Recall that symmetry operations in supmech (defined in section IV C) involve pairs of mappings $\Phi = (\Phi_1, \Phi_2)$ such that Φ_1 is a canonical transformation on the system algebra and $\Phi_2 = [\tilde{\Phi}_1]^{-1}$ is a transformation on states such that when both the mappings are applied, the expectation values are preserved. In a Hilbert space-based SQS, a symmetry operation must preserve transition probabilities (the condition defining a symmetry in Wigner's treatment [99] of symmetries in QM). Recalling Wigner's theorem (leading to implementation of quantum symmetries by unitary or antiunitary operators) and the fact that a symmetry operation on the system algebra of an SQS must be an automorphism of the algebra, we see that, while unitary transformations (mapping $\mathcal{D}$ onto itself – only these are permitted to represent quantum symmetries in our formalism) are genuine symmetry operations, the antiunitary transformations are not.

A symmetry implemented (in the unimodal sense, as defined in section IV C) by a unitary operator U acts on a state vector $\psi \in \mathcal{D}$ according to $\psi \to \psi' = U\psi$ and (when its action is transferred to operators) on an operator $A \in \mathcal{A}$ according to $A \to A'$ such that, for $\phi, \psi \in \mathcal{D}$,

$$(\phi', A\psi') = (\phi, A'\psi) \quad \Rightarrow A' = U^{-1}AU. \tag{166}$$

For an infinitesimal unitary transformation $U \simeq I + i\epsilon G$ where G is an even, Hermitian element of $\mathcal{A}$ [this follows from the condition $(U\phi, U\psi) = (\phi, \psi)$ for all $\phi, \psi \in \mathcal{D}$]. Considering the transformation $A \to A'$ in Eq.(166) as a quantum canonical transformation, generated (through PBs) by an element $T \in \mathcal{A}$, we have

$$\delta A = -i\epsilon[G, A] = \epsilon\{T, A\} \tag{167}$$

giving

$$T = -i(-i\hbar)G = -\hbar G \tag{168}$$

and

$$U \simeq I - i\frac{\epsilon}{\hbar}T. \tag{169}$$

It is the appearance of $\hbar$ in Eq(169) which is responsible for the appearance of the Planck constant at almost all conventional places in QM.

The quantum canonical transformation representing evolution in time of an SQS is implemented on the state vectors by a one-parameter family of unitary operators [in the form $\psi(t) = U(t-s)\psi(s)$] generated by the Hamiltonian operator H : $U(\epsilon) \simeq I - i\frac{\epsilon}{\hbar}H$. This gives, in the Schrödinger picture, the Schrödinger equation for the evolution of pure states of a Hilbert space based SQS :

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t). \tag{170}$$

In the Heisenberg picture, we have, of course, the Heisenberg equation of motion (164), which is now an operator equation.

Quantum triples provide a natural setting for a mathematically rigorous development of the Dirac bra-ket formalism [100]. The essence of this formalism lies in generalizing the orthogonal expansions in a Hilbert space to include integrals over 'generalized eigenvectors' (Fourier transforms, for example). This becomes necessary when some observables of interest have (partly or wholly) continuous spectrum. The appropriate formalism for this is provided by a 'rigged Hilbert space' (or Gel'fand triple) [101]; the latter appears as a natural development once a pair $(\mathcal{H}, \mathcal{D})$ consisting of a Hilbert space $\mathcal{H}$ and a dense linear subset $\mathcal{D}$ in it is given.

Given a dense domain $\mathcal{D}$ in $\mathcal{H}$, one can define the *-algebra $L^+(\mathcal{D})$ (in the notation of Lassner [52]) of all operators A such that both A and $A^\dagger$ are defined on $\mathcal{D}$ and map $\mathcal{D}$ into itself; the *-operation on $L^+(\mathcal{D})$ is given by $A^* = A^\dagger|\mathcal{D}$. A unital *-subalgebra of $L^+(\mathcal{D})$ is called an Op*-algebra. The algebra $\mathcal{A}$ of our quantum triple is clearly an Op*-algebra. [If, instead of choosing $\mathcal{A}$ first and then constructing $\mathcal{D}$, we had chosen $\mathcal{D}$ first and then proceeded to choose $\mathcal{A}$, then the natural/simplest choice of $\mathcal{A}$ would be $L^+(\mathcal{D})$.]

On $\mathcal{D}$, a locally convex topology t is defined by the seminorms $\|.\|_A$ given by

$$\|\psi\|_A = \|A\psi\|, \quad A \in L^+(\mathcal{D}); \tag{171}$$

we denote the resulting locally convex topological vector space by $\mathcal{D}[t]$. Let $\mathcal{D}'[t']$ be the dual space of $\mathcal{D}[t]$ with the strong topology [102] t' (it is defined by the seminorms

$$p_B(\phi) = sup_{\psi \in B} | < \phi, \psi > |$$

for all bounded subsetsB of $\mathcal{D}$. Then the Gelfand triple

$$\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}'[t']$$

constitutes the *canonical rigged Hilbert space* [52] based on the pair $(\mathcal{H}, \mathcal{D})$. The space $\mathcal{D}'[t']$ (the space of continuous linear functionals or distributions on the test function space $\mathcal{D}[t]$) is the space of bra vectors of Dirac. The space of kets is the space $\mathcal{D}^\times$ of continuous antilinear functionals on $\mathcal{D}[t]$. [An element $\chi \in \mathcal{H}$ defines a continuous linear functional F_χ and an antilinear functional K_χ on $\mathcal{H}$ (hence on $\mathcal{D}$) given by $F_\chi(\psi) = (\chi, \psi)$ and $K_\chi(\psi) = (\psi, \chi)$; both the bra and ket spaces, therefore, have $\mathcal{H}$ as a subset.]

A Hermitian operator A ($=A^*$) in $L^+(\mathcal{D})$ which admits a unique self adjoint extension in $\mathcal{H}$ (often called 'essentially self adjoint') and is cyclic [i.e. there exists a vector ψ in $\mathcal{D}$ such that the vectors $P(A)\psi$ where P(A) is a polynomial in A are dense in $\mathcal{H}$] has complete sets of generalized eigenvectors [eigenkets $\{|\lambda >; \lambda \in \sigma(A)$, the spectrum of A $\}$ and eigenbras $\{< \lambda|; \lambda \in \sigma(A)\}$] :

$$A|\lambda >= \lambda|\lambda >; \quad < \lambda|A = \lambda < \lambda|;$$

$$\int_{\sigma(A)} d\mu(\lambda)|\lambda >< \lambda| = I \tag{172}$$

where I is the unit operator in $\mathcal{H}$ and μ is a unique measure on $\sigma(A)$. These equations are to be understood in the sense that, for all $\phi, \psi \in \mathcal{D}$,

$$< \phi|A|\lambda >= \lambda < \phi|\lambda >; \quad < \lambda|A|\phi >= \lambda < \lambda|\phi >;$$

$$\int_{\sigma(A)} d\mu(\lambda) < \phi|\lambda >< \lambda|\psi >=< \phi|\psi > .$$

The last equation implies the expansion

$$|\psi >= \int_{\sigma(A)} d\mu(\lambda)|\lambda >< \lambda|\psi > .$$

More generally, one has complete sets of generalized eigenvectors associated with complete sets of commuting observables. For more details on the mathematically rigorous development of the bra-ket formalism, we refer to the literature [103-106].

C. Inevitability of the Hilbert space

Having shown the advantages of a Hilbert space-based realization of a standard quantum system, we now proceed to consider the existence and inevitability of such a realization.

Given an (abstract) SQS $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$, by a Hilbert space realization of it we mean an SQS $\hat{\Sigma} = (\hat{\mathcal{A}}, \hat{\mathcal{S}}_1, \hat{\omega}, \hat{H})$ of the type treated in the previous subsection which is equivalent to Σ as a supmech Hamiltonian system. This amounts to constructing a quantum triple $(\mathcal{H}, \mathcal{D}, \hat{\mathcal{A}})$ in which the algebra $\hat{\mathcal{A}}$ is isomorphic, as a topological algebra, to the system algebra $\mathcal{A}$ and choosing $\mathcal{D}$ to be maximal (in the sense of section VII B). Elements of $\mathcal{D}$ then provide pure states such that the observables (i.e. the Hermitian elements of $\hat{\mathcal{A}}$) and pure states satisfy the CC condition; moreover, once $\hat{\mathcal{A}}$ has been obtained (as a special algebra isomorphic to $\mathcal{A}$), construction of $\hat{\omega}$ and $\hat{H}$ is automatic.

From the above definition it is clear that, such a realization, if it exists, is unique upto equivalence.

Mathematically we have the problem of obtaining a faithful irreducible *-representation of the *-algebra $\mathcal{A}$. Good references for the treatment of relevant mathematical concepts are Powers [107] and Dubin and Hennings [51]. By a *-representation of a *-algebra $\mathcal{A}$ we mean a triple $(\mathcal{H}, \mathcal{D}, \pi)$ where $\mathcal{H}$ is a (separable) Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and π a *-homomorphism of $\mathcal{A}$ into the operator algebra $L^+(\mathcal{D})$ (defined earlier) satisfying the relation

$$(\chi, \pi(A)\psi) = (\pi(A^*)\chi, \psi) \quad \text{for all} \quad A \in \mathcal{A} \text{ and } \chi, \psi \in \mathcal{D}.$$

We shall build up our arguments such that no new assumptions will be involved in going from the abstract algebraic setting to the Hilbert space setting; emergence of the Hilbert space formalism will be automatic.

To this end, we shall exploit the fact that the CC condition guarantees the existence of plenty of (pure) states of the algebra $\mathcal{A}$. Given a state ϕ on $\mathcal{A}$, a standard way to obtain a representation of $\mathcal{A}$ is to employ the so-called GNS construction. Some essential points related to this construction are given below :

(i) Noting that the given algebra $\mathcal{A}$ is itself a complex vector space, one tries to define a scalar product on it using the state ϕ, the obvious choice being $(A, B) = \phi(A^*B)$. This, however, is prevented from being positive definite if the set

$$L_\phi = \{A \in \mathcal{A}; \phi(A^*A) = 0\}$$

(which can be shown to be a left ideal of $\mathcal{A}$) has nonzero elements in it. On the quotient space $\mathcal{D}_\phi^{(0)} = \mathcal{A}/L_\phi$, the object

$$([A], [B]) = \phi(A^*B) \tag{173}$$

is a well defined scalar product. Here $[A] = A + L_\phi$ denotes the equivalence class of A in ${\mathcal{D}^{(0)}}_\phi$.
(ii) One then completes the inner product space $(\mathcal{D}_\phi^{(0)}, (,))$ to obtain the Hilbert space $\mathcal{H}_\phi$; it is guaranteed to be separable by the separability of the topological algebra $\mathcal{A}$.
(iii) One defines a representation $\pi_\phi^{(0)}$ of $\mathcal{A}$ on the pair $(\mathcal{H}_\phi, \mathcal{D}_\phi^{(0)})$ by

$$\pi_\phi^{(0)}(A)[B] = [AB]; \tag{174}$$

it can be easily checked to be a well defined *-representation.
(iv) The operators $\pi_\phi^{(0)}(A)$ induce a topology on $\mathcal{D}_\phi^{(0)}$ [see Eq.(171)]; the completion $\mathcal{D}_\phi$ of $\mathcal{D}_\phi^{(0)}$ in this topology acts as the common invariant domain for the operators $\pi_\phi(A)$ (where π_ϕ is the closure of the representation $\pi_\phi^{(0)}$).
(v) The original state ϕ is represented as a vector state in the representations $\pi_\phi^{(0)}$ and π_ϕ by the vector $\chi_\phi = [I]$ (the equivalence class of the unit element of $\mathcal{A}$); indeed, we have, from Eq.(173),

$$\begin{aligned} \phi(A) &= ([I], [A]) = ([I], \pi_\phi^{(0)}(A)[I]) \\ &= (\chi_\phi, \pi_\phi^{(0)}(A)\chi_\phi) = (\chi_\phi, \pi_\phi(A)\chi_\phi). \end{aligned} \tag{175}$$

(vi) The triple $(\mathcal{H}_\phi, \mathcal{D}_\phi, \pi_\phi)$ satisfying Eq.(175), referred to as the GNS representation of $\mathcal{A}$ induced by the state ϕ [some authors refer to the triple $(\mathcal{H}_\phi, \mathcal{D}_\phi^{(0)}, \pi_\phi^{(0)})$ as the GNS representation of $\mathcal{A}$], is determined uniquely, up to unitary equivalence, by the state ϕ.
(vii) The representation π_ϕ of $\mathcal{A}$ is irreducible if and only if the state ϕ is pure.

This construction, however, does not completely solve our problem because a GNS representation is generally not faithful; for all $A \in L_\phi$, we have obviously $\pi_\phi(A) = 0$. [For example, a state with zero expectation value for the kinetic energy of a particle will yield a GNS representation which will represent the momentum observable of the particle by the zero operator.]

Note. The GNS representation is faithful if the state ϕ is faithful (i.e. if $L_\phi = \{0\}$). Such a state, however, is not guaranteed to exist by our postulates.

A faithful but generally reducible representation of $\mathcal{A}$ can be obtained by taking the direct sum of the representations of the above sort corresponding to *all* the pure states ϕ. (For the construction of the direct sum of a possibly uncountable set of Hilbert spaces, see Ref.[108].)

Let $\mathcal{K}$ be the Cartesian product of the Hilbert spaces $\{\mathcal{H}_\phi : \phi \in \mathcal{S}_1(\mathcal{A})\}$. A general element ψ of $\mathcal{K}$ is a collection $\{\psi_\phi : \phi \in \mathcal{S}_1(\mathcal{A})\}$; here ψ_ϕ is called the component of ψ in $\mathcal{H}_\phi$. The desired Hilbert space $\mathcal{H}$ consists of those elements ψ in $\mathcal{K}$ which have an at most countable set of nonzero components ψ_ϕ satisfying the condition

$$\sum_\phi \|\psi_\phi\|^2_{\mathcal{H}_\phi} < \infty.$$

The scalar product in $\mathcal{H}$ is given by

$$(\psi, \psi') = \sum_\phi (\psi_\phi, \psi'_\phi)_{\mathcal{H}_\phi}.$$

The direct sum of the representations $\{(\mathcal{H}_\phi, \mathcal{D}_\phi, \pi_\phi); \phi \in \mathcal{S}_1(\mathcal{A})\}$ is the representation $(\mathcal{H}, \mathcal{D}, \pi)$ where $\mathcal{H}$ is as above, $\mathcal{D}$ is the subset of $\mathcal{H}$ consisting of vectors ψ with $\psi_\phi \in \mathcal{D}_\phi$ for all $\phi \in \mathcal{S}_1(\mathcal{A})$ and, for any $A \in \mathcal{A}$,

$$\pi(A)\psi = \{\pi_\phi(A)\psi_\phi; \phi \in \mathcal{S}_1(\mathcal{A})\}.$$

Now, given any two different elements A_1, A_2 in $\mathcal{A}$, let ϕ_0 be a pure state (guaranteed to exist by the CC condition) such that $\phi_0(A_1) \neq \phi_0(A_2)$. Let $\psi_0 \in \mathcal{H}$ be the vector with the single nonzero component χ_{ϕ_0}. For any $A \in \mathcal{A}$, we have

$$(\psi_0, \pi(A)\psi_0) = (\chi_{\phi_0}, \pi_{\phi_0}(A)\chi_{\phi_0}) = \phi_0(A).$$

This implies

$$(\psi_0, \pi(A_1)\psi_0) \neq (\psi_0, \pi(A_2)\psi_0), \quad \text{hence } \pi(A_1) \neq \pi(A_2)$$

showing that the representation $(\mathcal{H}, \mathcal{D}, \pi)$ is faithful.

The Hilbert space $\mathcal{H}$ obtained above may be non-separable (even if the spaces $\mathcal{H}_\phi$ are separable); this is because the set $\mathcal{S}_1(\mathcal{A})$ is generally uncountable. To obtain a faithful representation of $\mathcal{A}$ on a separable Hilbert space, we shall use again the separability of $\mathcal{A}$ as a topological algebra. Let $\mathcal{A}_0 = \{A_1, A_2, A_3, ...\}$be a countable dense subset of $\mathcal{A}$ consisting of nonzero elements. The CC condition allows us to find pure states ϕ_j (j=1,2,...) such that

$$\phi_j(A_j^* A_j) \neq 0, \quad j = 1, 2, ... \tag{176}$$

Now consider the GNS representations $(\mathcal{H}_{\phi_j}, \mathcal{D}_{\phi_j}, \pi_{\phi_j})$ (j=1,2,...). Eq.(176) guarantees that

$$\pi_{\phi_j}(A_j) \neq 0, \quad j = 1, 2, ... \tag{177}$$

Indeed

$$\begin{aligned} 0 \neq \phi_j(A_j^* A_j) &= (\chi_{\phi_j}, \pi_{\phi_j}(A_j^* A_j)\chi_{\phi_j}) \\ &= (\pi_{\phi_j}(A_j)\chi_{\phi_j}, \pi_{\phi_j}(A_j)\chi_{\phi_j}). \end{aligned}$$

Now consider the direct sum $(\mathcal{H}', \mathcal{D}', \pi')$ of these representations. To show that π' is faithful, we must show that, for any nonzero element A of $\mathcal{A}$, $\pi'(A) \neq 0$. This is guaranteed by Eq.(177) because, $\mathcal{A}_0$ being dense in $\mathcal{A}$, A can be arranged to be as close as we like to some A_j in $\mathcal{A}_0$.

The representation π', however, is in general reducible. To obtain a faithful irreducible representation, we should try to obtain the relations $\pi(A_j) \neq 0$ (j= 1,2,..) in a single representation π. To this end, let $B^{(k)} = A_1 A_2 ... A_k$ and choose $\phi^{(k)} \in \mathcal{S}_1(\mathcal{A})$ such that

$$\phi^{(k)}(B^{(k)*} B^{(k)}) \neq 0.$$

In the GNS representation $(\mathcal{H}_{\phi^{(k)}}, \mathcal{D}_{\phi^{(k)}}, \pi_{\phi^{(k)}})$, we have

$$0 \neq \pi_{\phi^{(k)}}(B^{(k)}) = \pi_{\phi^{(k)}}(A_1) ... \pi_{\phi^{(k)}}(A_k)$$

which implies

$$\pi_{\phi^{(k)}}(A_j) \neq 0, \quad j = 1, ..., k. \tag{178}$$

This argument works for arbitrarily large but finite k. If the $k \to \infty$ limit of the above construction leading to a limiting GNS representation $(\underline{\mathcal{H}}, \underline{\mathcal{D}}, \underline{\pi})$ exists, giving

$$\underline{\pi}(A_j) \neq 0, \quad j = 1, 2, ..., \tag{179}$$

then, by an argument similar to that for π' above, one must have $\underline{\pi}(A) \neq 0$ for all non-zero A in $\mathcal{A}$ showing faithfulness of $\underline{\pi}$.

Note. (i) For finitely generated system algebras (this covers all applications of QM in atomic physics), a limiting construction is not needed; the validity of Eq.(178) for sufficiently large k is adequate. [Hint : Take the generators of the algebra $\mathcal{A}$ as elements of $\mathcal{A}_0$.]
(ii) For general algebras, it appears that some extra condition is needed to arrive at a faithful irreducible representation.
(iii) The developments in this subsection did not require the algebra $\mathcal{A}$ to be special; the results obtained are, therefore, valid for more general quantum systems. We shall use this fact in section VII E.
(iv) In fact, even non-commutativity of the algebra $\mathcal{A}$ was not used above. This, however, is not surprising; commutative algebras, under fairly general conditions, can be realized as algebras of operators in Hilbert spaces.

A more complete treatment of these matters is intended to be presented when the treatment of quantum field theory in an appropriately augmented supmech framework is taken up.

Having shown the existence and desirability of the Hilbert space-based realizations for finitely generated system algebras, we now have a formal justification for the direct route to the Hilbert space taken in the traditional treatment of QM of localizable elementary systems (massive particles), namely, employment of irreducible unitary representations of the effective relativity group $\hat{G}_0$ (or, equivalently, projective unitary representations of the relativity group G_0). This is the simplest way to satisfy the condition of transitive action of $\hat{G}_0$ on the space of pure states and simultaneously satisfy the CC condition.

We take up the QM of these objects in the next subsection.

D. Quantum mechanics of localizable elementary systems (massive particles)

A *quantum elementary system* is an SQS which is also an elementary system. The concept of a quantum elementary system, therefore, combines the concept of quantum symplectic structure with that of a relativity scheme. The basic entities relating to an elementary system are its fundamental observables which generate the system algebra $\mathcal{A}$. For quantum elementary systems, this algebra $\mathcal{A}$ has the quantum symplectic structure as described in subsection A above. All the developments in section V can now proceed with the PBs understood as quantum PBs. We shall employ the Hilbert space-based realizations of these systems.

The effective relativity group $\hat{G}_0$ has a Poisson action on $\mathcal{A}$ and a transitive action on the set $\mathcal{S}_1(\mathcal{A})$ of pure states of $\mathcal{A}$. We have seen in section VII B that, in a Hilbert space based realization of an SQS in terms of a quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$, a symmetry operation can be represented as a unitary operator on $\mathcal{H}$ mapping $\mathcal{D}$ onto itself. A symmetry group is then realized as a unitary representation on $\mathcal{H}$ such that the representative operators map $\mathcal{D}$ onto itself. For an elementary system the condition of transitive action on $\mathcal{S}_1$ implies that this representation must be irreducible.(There is no contradiction between this requirement and that of invariance of $\mathcal{D}$ because $\mathcal{D}$ is not a closed subspace of $\mathcal{H}$ when $\mathcal{H}$ is infinite dimensional.)

By a (quantum) *particle* we shall mean a localizable (quantum) elementary system. We shall first consider nonrelativistic particles. The configuration space of a nonrelativistic particle is the 3-dimensional Euclidean space R^3. The fundamental observables for such a system were identified, in section V B, as the mass (m) and Cartesian components of position (X_j), momentum (P_j) and spin(S_j) (j = 1,2,3) satisfying the PB relations in equations (107,108,102). The mass m will be treated, as before, as a positive parameter. The system algebra $\mathcal{A}$ of the particle is the *-algebra generated by the fundamental observables (taken as hermitian) and the unit element. Since it is an ordinary *-algebra (i.e. one not having any fermionic objects), the supercommutators reduce to ordinary commutators. The PBs mentioned above now take the form of the commutation relations

$$[X_j, X_k] = 0 = [P_j, P_k], \quad [X_j, P_k] = i\hbar\delta_{jk} I \qquad (A)$$

$$[S_j, S_k] = i\hbar\epsilon_{jkl} S_l, \quad [S_j, X_k] = 0 = [S_j, P_k]. \qquad (B) \qquad (180)$$

We shall first consider the spinless particles ($\mathbf{S} = 0$). We, therefore, need to consider only the Heisenberg comutation relations (180A)[often referred to as the *canonical commutation relations*(CCR)]. Assuming the existence of a quantum triple $(\mathcal{H}, \mathcal{D}, \hat{\mathcal{A}})$ corresponding to this SQS, we shall employ some results obtained in section VII B to obtain the explicit construction. Here $\hat{\mathcal{A}}$ is the algebra generated by (representatives of) the fundamental observables X_j, P_j (j = 1,2,3) and the unit element I subject to the commutation relations (180A) and the pair $(\mathcal{H}, \mathcal{D})$ carries a faithful irreducible representation of the system algebra as explained above. We introduce the bra and ket spaces as in section VII B. Let $x = (x_1, x_2, x_3), dx = dx_1 dx_2 dx_3$ and $|x>, <x|$ the simultaneous eigenkets and eigenbras of the operators X_j (j= 1,2,3):

$$X_j|x> = x_j|x>, \quad <x|X_j = <x|x_j, \quad x_j \in R, \quad j = 1, 2, 3; \qquad (181)$$

they are assumed (with a promise of justification later) to form a complete set providing a resolution of identity in the form

$$I = \int_{R^3} |x > dx < x|. \tag{182}$$

Given any vector $|\psi > \in \mathcal{D}$, the corresponding wave function $\psi(x) =< x|\psi >$ must satisfy the relation

$$(X_j\psi)(x) =< x|X_j|\psi >= x_j\psi(x). \tag{183}$$

Recalling the discussion of localization in section IV G, the localization observable P(D) corresponding to a Borel set D in R^3 is represented as the operator

$$P(D) = \int_D |x > dx < x|. \tag{184}$$

[The required properties of P(D) are easily verified.] Given the particle in the state $|\psi > \in \mathcal{D}$, the probability that it will be found in the domain D is given by

$$< \psi|P(D)|\psi >= \int_D < \psi|x > dx < x|\psi >= \int_D |\psi(x)|^2 dx \tag{185}$$

giving the traditional Born interpretation of the wave function ψ. The integral above is meaningful for all Borel sets D only if ψ is square integrable over R^3 which implies $\mathcal{H} = L^2(R^3, dx)$.

To determine the operators P_j, we must choose the unitary operators $U(a)$ representing space translations such that the infinitesimal generators satisfy the last two equations in (180A). The simplest choice for $U(a)$, namely,

$$[U(a)\psi](x) = \psi(x - a)$$

[which is a special case of of the relation $[U(g)\psi](x) = \psi(T_g^{-1}x)$; these operators are unitary when the transformation T_g preserves the Lebesgue measure on R^3] happens to be adequate. Recalling Eq.(169), we have, for an infinitesimal translation,

$$\delta\psi = -\frac{i}{\hbar}\mathbf{a}.\mathbf{P}\psi = -\mathbf{a}.\bigtriangledown\psi$$

giving the operators P_j representing momentum components as

$$(P_j\psi)(x) = -i\hbar\frac{\partial\psi}{\partial x_j} \tag{186}$$

which satisfy the desired commutation relations.

We now identify the space $\mathcal{D}$ as [51]

$$\mathcal{D} = C^\infty(X_j, P_j; j = 1, 2, 3) = \mathcal{S}(R^3).$$

The operators U(a) clearly map this domain onto itself. With this choice of $\mathcal{D}$, the operators X_j and P_j given by equations (183) and (186) are essentially self adjoint. The two triples of

operators $\{X_j\}$ and $\{P_j\}$ separately constitute complete sets of commuting operators. The completeness of the X_js can be easily seen by (operating in the X-representation and) taking the harmonic oscillator ground state wave function as the cyclic vector. Similar argument works for the P_js in the momentum representation. This confirms the legitimacy of the Dirac constructions employed above.

The pair $(\mathcal{H}, \mathcal{D}) = (L^2(R^3), \mathcal{S}(R^3))$ with operators X_j and P_j as constructed above is known as the *Schrödinger representation* of the CCR (180A).

Closures of the operators P_j, X_j (which are self adjoint and are denoted by the same symbols) generate the unitary groups of operators $U(a) = exp(-ia.P)$ and $V(b) = exp(-ib.X)$ (where $a.P = \sum_j a_j P_j$ etc. and we have put $\hbar = 1$.) which satisfy the Weyl commutation relations

$$\begin{aligned} U(a)U(b) &= U(b)U(a) = U(a+b),\ V(a)V(b) = V(b)V(a) = V(a+b) \\ U(a)V(b) &= e^{ia.b}V(b)U(a). \end{aligned} \tag{187}$$

For all $\psi \in \mathcal{D}$, we have

$$(U(a)\psi)(x) = \psi(x-a), \quad (V(b)\psi)(x) = e^{-ib.x}\psi(x); \tag{188}$$

this is referred to as the Schrödinger representation of the Weyl commutation relations. According to the uniqueness theorem [47] of von Neumann, the irreducible representation of the Weyl commutation relations is, upto unitary equivalence, uniquely given by the Schrödinger representation (188).

Note. (i) Not every representation of the CCR (180A) with essentially self adjoint X_j and P_j gives a representation of the Weyl commutation relation. [For a counterexample, see Ref.[53], example (4.3.3).] A necessary and sufficient condition for the latter to materialize is that the harmonic oscillator Hamiltonian operator $H = P^2/(2m) + kX^2/2$ be essentially self adjoint. In the Schrödinger representation of the CCR obtained above, this condition is satisfied [14,51].

(ii) The von Neumann uniqueness theorem serves to confirm/verify, in the present case, the uniqueness (up to equivalence) of the Hilbert space realization of an SQS mentioned in section VII B.

Quantum dynamics of a free nonrelativistic spinless particle is governed, in the Schrödinger picture, by the Schrödinger equation (170) with $\psi \in \mathcal{D} = \mathcal{S}(R^3)$ and with the Hamiltonian (110)[where $\mathbf{P}$ is now the operator in Eq.(186)]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\bigtriangledown^2\psi \tag{189}$$

Explicit construction of the projective unitary representation of the Galilean group G_0 in the Hilbert space $\mathcal{H} = L^2(R^3, dx)$ and Galilean covariance of the free particle Schrödinger equation (189) have been treated in the literature [109,47,110].

When external forces are acting, the Hamiltonian operator has the more general form (111). Restricting V in this equation to a function of $\mathbf{X}$ only (as is the case in common applications), we thus obtain the traditional Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}\bigtriangledown^2 + V(\mathbf{X})]\psi. \tag{190}$$

Note. In Ref.[2], the free particle Hamiltonian operator of Eq.(127)(there) was arrived at by direct reasoning from the commutation relations of the projective Galilean group. (This has some instruction value.) Here we have used essentially similar reasoning to arrive at the free particle Hamiltonian of Eq.(108) in the more general context of supmech. In Ref.[2], the full Hamiltonian of Eq.(128)(there) has V(X) [instead of the V(X,P) of Eq.(109) above]. This is due to a mistake in the last stage of the argument there (which occurred while using the relation $[H', X_k] = 0$ which need not be valid when interactions are present.)

It should be noted that, in the process of obtaining the Schrödinger equation (190) for a nonrelativistic spinless particle with the traditional Hamiltonian operator, we did not use the classical Hamiltonian or Lagrangian for the particle. No *quantization* algorithm has been employed; the development of the quantum mechanical formalism has been autonomous, as promised.

From this point on, the development of QM along the traditional lines can proceed.

For nonrelativistic particles with $m > 0$ and $s \geq 0$, we have $\mathcal{H} = L^2(R^3, C^{2s+1})$ and $\mathcal{D} = \mathcal{S}(R^3, C^{2s+1})$. The treatment of spin being standard, we skip the details.

Relativistic elementary quantum systems have been treated extensively in literature [96,111,47,88a,97]. These treatments employ projective irreducible representations of the Poincaré group which can be obtained from the irreducible unitary representations of its covering group. The justification for this already has been given in the supmech formalism in the last para of section VII C. We shall skip the details.

E. Quantum systems with more general system algebras; Superselection rules

Now we consider general quantum systems which, as already defined in section VI B, are those with (not necessarily special) non-supercommutative system algebras equipped with quantum symplectic structure. Standard quantum systems are the subclass of these in which the system algebra $\mathcal{A}$ has a trivial graded center and only inner superderivations. We shall now relax these two conditions. On a noncommutative (super-)algebra $\mathcal{A}$ having a trivial graded center but having both inner and outer (super-)derivations, a quantum symplectic structure can be defined by employing the generalization of the supmech formalism treated in section IV F and operate with the generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega_Q)$ where $\mathcal{X}$= ISDer($\mathcal{A}$) (this gives, again, the quantum PBs of section VII A). For convenience of reference, we shall call this class of systems *quasi-standard quantum systems*.

We next consider the generalization involving a nontrivial graded center. We shall restrict ourselves to considering a nontrivial center only. The center $\mathcal{C} \equiv Z_0(\mathcal{A})$ of $\mathcal{A}$ is a commutative

locally convex algebra. Keeping in view that a faithful (not necessarily irreducible) Hilbert space-based realization of the superalgebra $\mathcal{A}$ is always possible, we shall operate, in this subsection, in the framework of a quantum triple $(\mathcal{H}, \mathcal{D}, \hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is a faithful realization of $\mathcal{A}$.

A nontrivial center $\mathcal{C}$ implies the presence of superselection rules and/or external fields. The two are, in fact, related : values of the external fields define superselection rules. Before taking up the general case, we consider a couple of illustrative situations :
(i) Consider first the situation when $\mathcal{C}$ is generated by a finite number of self-adjoint operators Q_s (s=1,..,n) each of which has a discrete spectrum (we shall call such observables *charge type observables*). In this case, we have $\mathcal{H} = \oplus_i \mathcal{H}_i$ where the spaces $\mathcal{H}_i$ are simultaneous eigenspaces of the observables Q_s. Defining $\mathcal{D}_i = \mathcal{D} \mid \mathcal{H}_i$ and $\mathcal{A}_i = \mathcal{A} \mid \mathcal{H}_i$, the algebras $\mathcal{A}_i$ have trivial center and the quantum triples $(\mathcal{H}_i, \mathcal{D}_i, \mathcal{A}_i)$ correspond to quasi-standard quantum systems; the operators Q_s act as superselection operators and the spaces $\mathcal{H}_i$ as coherent subspaces labeled by the set of eigenvalues of the superselection operators.
(ii) Functions representing (components of) external fields belong to the commutative algebra of functions on the space-time manifold; in the relevant situations, this algebra obviously belongs to the center of the system algebra.

Regarding the general situation, we note that generalizations of the famous Gel'fand-Naĭmark theorem on commutative C^*-algebras [9] to some classes of commutative locally convex *-algebras have appeared in literature [59,112]. These generalizations relate (through isomorphisms or, more generally, homomorphisms) the latter algebras to those of continuous functions on reasonably 'nice' classes of topological spaces (typically Tychonoff spaces [112]); these topological spaces will be referred to as the spectral spaces of the respective algebras. In the above-mentioned Hilbert space-based realization, the center $\mathcal{C}$ will be represented as a commutative algebra of operators with $\mathcal{D}$ as a common invariant domain. Generalization of the spectral theorem to commutative algebras of operators leads to a representation of $\mathcal{H}$ as a direct integral

$$\mathcal{H} = \int_{\Sigma}^{\oplus} \mathcal{H}(\lambda) d\sigma(\lambda) \tag{191}$$

where Σ is the spectral space of $\mathcal{C}$ and σ is a measure uniquely determined by the algebra $\mathcal{C}$. Defining $\mathcal{D}(\lambda)$ and $\hat{\mathcal{A}}(\lambda)$ as the restrictions of $\mathcal{D}$ and $\hat{\mathcal{A}}$ to $\mathcal{H}(\lambda)$, we have the quantum triples $(\mathcal{H}(\lambda), \mathcal{D}(\lambda), \hat{\mathcal{A}}(\lambda))$ representing quasi-standard quantum systems. The Hilbert spaces $\mathcal{H}(\lambda)$ are traditionally referred to as coherent subspaces.

Generally the space Σ will be disconnected; the integral (191) will then reduce to a sum of integrals of the same type over the connected pieces of Σ. The examples (i) and (ii) above correspond to the two extreme situations when the space Σ is, respectively, discrete and connected (in the latter case a space-time domain).

An important class of examples corresponds to the situation when there are one or more mutually commuting operators with continuous spectra defining superselection rules. The mass

operator of a Galilean particle is a good example. Had we not invoked, in section V B, the CC condition to have M = mI and had treated the mass observable M simply as an element of the center $\mathcal{C}$ (thereby making the latter non-trivial), we could not have treated the system in question as a standard quantum system. Instead, we would have the situation in Eq.(191) with λ replaced by the mass parameter m and and Σ an interval of the form $[a, \infty)$ with $a > 0$. Galilean relativity provides no clue to the value of a; it may tentatively be taken to be the mass of the least massive among the positive mass particles in nature.

A more systematic treatment of the general case is expected to be presented in a future work relating to a supmech-based treatment of quantum field theory.

Before closing this subsection, it is worth emphasizing that

(i) in the present formalism, there is a natural place for superselection rules, and
(ii) the superselection rules arising as described above are commutative — a highly desirable feature.

For a somewhat complementary treatment of matters related to this subsection, we refer to the insightful paper of Jauch and Misra [113].

F. Classical Systems

Continuing the treatment, in section III H, of classical symplectic structures as special cases of the symplectic structures of section III F, we note here that a classical Hamiltonian system (M, ω_{cl}, H_{cl}) is realized in supmech as the Hamiltonian system $(\mathcal{A}_{cl}, \omega_{cl}, H_{cl})$ where $\mathcal{A}_{cl} = C^\infty(M)$ and H_{cl} a smooth real-valued function which is bounded below. The supmech Hamilton equation (64) is, in the present context, nothing but the traditional Hamilton equation:

$$\frac{df}{dt} = \{H_{cl}, f\}_{cl}. \tag{192}$$

States in the present context are probability measures on M; in obvious notation, they are of the form

$$\phi_\mu(f) = \int_M f d\mu. \tag{193}$$

Pure states are Dirac measures (or, equivalently, points of M) $\mu_{\xi_0} (\xi_0 \in M)$ for which $\phi_{\xi_0}(f) = f(\xi_0)$. The pair $(\mathcal{O}(\mathcal{A}_{cl}), \mathcal{S}_1(\mathcal{A}_{cl}))$ of classical observables and pure states is easily seen to satisfy the CC condition : Given two different real-valued smooth functions on M, there is a point of M at which they take different values; conversely, given two different points of M, there is a real-valued smooth function on M which takes different values at those points.

In ordinary mechanics, only pure states are used. Expectation values of *all* observables in these states are their precise values at the relevant points and the theory is deterministic (obtained here as a special case of a probabilistic theory). More general states are employed in classical statistical mechanics where, in most applications, they are taken to be represented by densities on M [$d\mu = \rho(\xi) d\xi$ where $d\xi = dqdp$ is the Liouville volume element on M].

The state evolution equation (65) of supmech gives, in the present context

$$\int_M (\frac{\partial \rho(\xi,t)}{\partial t})(\xi) f(\xi) d\xi = \int_M \rho(\xi,t)\{H,f\}_{cl}(\xi) d\xi. \tag{194}$$

For the remainder of this subsection, we take $M = R^{2n}$. To satisfy the normalization condition, the density ρ must vanish at infinity. Now, using Eq.(46) for the Poisson bracket in Eq.(194), performing a partial integration and discarding the (vanishing) surface term, the right hand side of Eq.(194) becomes

$$-\int_M \omega_{cl}^{ab}(\xi) \frac{\partial H}{\partial \xi^a} \frac{\partial \rho}{\partial \xi^b} f d\xi = \int_M \{\rho, H\}_{cl} f d\xi.$$

Since f is arbitrary, Eq.(194) now gives the traditional Liouville equation

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}_{cl}. \tag{195}$$

A classical Galilean elementary system is a system characterized/labelled by the three Galilean invariants m,s,u. Its fundamental observables other than the invariants are the position, momentum and spin vectors **X,P,S** satisfying the PB relations of section VB (where the symbols now represent phase space variables). The observable **X** has the interpretation of the position vector of the center of mass of the system. A particle is an elementary system with the internal energy u = 0 and negligible size so that **X** now refers to the particle position. The free particle Hamiltonian for for a spinless particle is given by Eq.(110) and the one with interaction in Eq.(111). For a detailed treatment of classical Galilean systems we refer to the literature [85,88]. The treatment of classical spin in the framework of sphere S^2 as the symplectic manifold may be found in [94].

G. Superclasical systems

Superclassical mechanics is an extension of classical mechanics which employs, besides the traditional phase space variables, Grassmann variables $\theta^\alpha (\alpha = 1, ..n$, say) satisfying the relations

$$\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0 \quad \text{for all } \alpha, \beta;$$

[in particular $(\theta^\alpha)^2 = 0$ for all α]. These objects generate the so -called Grassmann algebra (with n generators) $\mathcal{G}_n$ whose elements are functions of the form

$$f(\theta) = a_0 + a_\alpha \theta^\alpha + a_{\alpha\beta}\theta^\beta \theta^\alpha + ... \tag{196}$$

where the coefficients $a_{..}$ are complex numbers. If the coefficients in Eq.(196) are taken to be smooth functions on, say, R^m, the resulting functions $f(x,\theta)$ are referred to as smooth functions on the superspace $R^{m|n}$; the algebra of these functions is denoted as $C^\infty(R^{m|n})$. With parity zero assigned to the variables x^a (a = 1,..,m) and one to the $\theta^\alpha, C^\infty(R^{m|n})$ is a supercommutative superalgebra. Restricting the variables x^a to an open subset U of R^m,

one obtains the superdomain $U^{m|n}$ and the superalgebra $C^\infty(U^{m|n})$ in the above-mentioned sense. Gluing such superdomains appropriately, one obtains the objects called supermanifolds [72,71,114,115]. These are the objects serving as phase spaces in superclassical mechanics. We shall, for simplicity, restrict ourselves to the simplest supermanifolds $R^{m|n}$ and take, for the development of supermechanics in the present context, $\mathcal{A} = C^\infty(R^{m|n})$. The 'coordinate variables' x^a, θ^α will be jointly referred to as ξ^A. We shall write $\epsilon(\xi^A) = \epsilon_A$. A *-operation is assumed to be defined on $\mathcal{A}$ for which $(\xi^A)^* = \xi^A$.

It is useful to define left and right differentiations with respect to the odd variables as follows (the subscripts l and r refer to left and right)

$$\frac{\partial_l}{\partial\theta^\alpha}(\theta^{\alpha_1}..\theta^{\alpha_s}) = \delta_\alpha^{\alpha_1}\theta^{\alpha_2}..\theta^{\alpha_s} - \delta_\alpha^{\alpha_2}\theta^{\alpha_1}\theta^{\alpha_3}..\theta^{\alpha_s} + .. \\ (-1)^{s-1}\delta_\alpha^{\alpha_s}\theta^{\alpha_1}..\theta^{\alpha_{s-1}} \tag{197}$$

$$\frac{\partial_r}{\partial\theta^\alpha}(\theta^{\alpha_1}..\theta^{\alpha_s}) = \delta_\alpha^{\alpha_s}\theta^{\alpha_1}..\theta^{\alpha_{s-1}} - \delta_\alpha^{\alpha_{s-1}}\theta^{\alpha_1}..\theta^{\alpha_{s-2}}\theta^{\alpha_s} + .. \\ (-1)^{s-1}\delta_\alpha^{\alpha_1}\theta^{\alpha_2}..\theta^{\alpha_s} \tag{198}$$

and extend by linearity to general elements of $\mathcal{A}$. Taking

$$\frac{\partial_l f}{\partial x^a} = \frac{\partial_r f}{\partial x^a} \equiv \frac{\partial f}{\partial x^a},$$

we now have left and right derivatives with respect to ξ^A defined on $\mathcal{A}$. Defining

$${}_Ae = \frac{\partial_l}{\partial\xi^A}, \qquad e_A = \frac{\partial_r}{\partial\xi^A}$$

we have, for any two homogeneous elements f,g of $\mathcal{A}$,

$$ {}_Ae(fg) = ({}_Aef)g + (-1)^{\epsilon_f\epsilon_A}f({}_Aeg); \\ e_A(fg) = f(e_Ag) + (-1)^{\epsilon_g\epsilon_A}(e_Af)g. \tag{199}$$

The objects ${}_Ae$ (but not e_A) are superderivations of the superalgebra $\mathcal{A}$. A general element of $SDer(\mathcal{A})$ (called a supervectorfield) is of the form

$$X = X^A(\xi)\ {}_Ae = e_A\ {}^AX(\xi). \tag{200}$$

The differential of a function $f \in \mathcal{A}$ can be written as

$$df = d\xi^A\frac{\partial_l f}{\partial\xi^A} = \frac{\partial_r f}{\partial\xi^A}d\xi^A \tag{201}$$

where $d\xi^A$ are symbols serving as basis vectors in the space of 1-forms. We have

$$\frac{\partial_r f}{\partial\xi^A} = (-1)^{\epsilon_A(\epsilon_f+\epsilon_A)}\frac{\partial_l f}{\partial\xi^A}. \tag{202}$$

A general 1-form $\omega^{(1)}$ and a 2-form $\omega^{(2)}$ can be written as

$$\omega^{(1)} = \omega^{(1)}_A d\xi^A = d\xi^A \ {}_A\omega^{(1)} \tag{203}$$

$$\omega^{(2)} = \omega^{(2)}_{AB} d\xi^B d\xi^A = d\xi^B d\xi^A \ {}_{AB}\omega = d\xi^A \ {}_A\omega_B d\xi^B. \tag{204}$$

Note that, when A,B are odd, $\omega^{(2)}_{AB} = \omega^{(2)}_{BA}$. It follows that, the odd dimension n (in $R^{m|n}$) need not be an even number for a symplectic form to exist; the number m must, of course, be even.

Given a symplectic form ω on $\mathcal{A}$, we have, for any $f \in \mathcal{A}$,

$$\omega(X_f, Y) = -(df)(Y) \tag{205}$$

where X_f is the Hamiltonian supervector field corresponding to f and $Y \in SDer(\mathcal{A})$. Since ω is even, Equations (204,205) give (writing $f_{,A}$ for $\frac{\partial_r f}{\partial \xi^A}$)

$$X^A_f \, {}_A\omega_B = -f_{,B}\,. \tag{206}$$

On $R^{m|n}$, the symplectic form can be chosen so that the coefficients ${}_A\omega_B$ are independent of ξ. Assuming this and introducing the inverse $({}^A\omega^B)$ of the matrix $({}_A\omega_B)$: ${}_A\omega_B \ {}^B\omega^C = \delta^C_A$, we have

$$X^A_f = -f_{,B} \ {}^B\omega^A. \tag{207}$$

The Poisson bracket of $f, g \in \mathcal{A}$ is

$$\{f, g\} = X_f(g) = X^A_f \frac{\partial_l g}{\partial \xi^A} = -\frac{\partial_r f}{\partial \xi^B} \ {}^B\omega^A \frac{\partial_l g}{\partial \xi^A}. \tag{208}$$

The dynamics is governed by the supmech Hamilton equation (64) with H an even Hermitian element of $\mathcal{A}$ and the Poisson bracket of Eq.(208).

States in superclassical mechanics are linear functionals on $\mathcal{A} = C^\infty(R^{m|n})$; they are generalizations of the states (193) given by

$$\phi(f) = \int_{R^{m|n}} f(x,\theta) d\mu(x,\theta) \tag{209}$$

where the measure μ satisfies the normalization and positivity conditions

$$1 = \phi(1) = \int d\mu(x,\theta) \tag{210}$$

$$0 \leq \int f f^* d\mu \quad \text{for all } f \in \mathcal{A}. \tag{211}$$

In the rest of this subsection, we shall consider only states represented by a density function :

$$d\mu(x,\theta) = \rho(x,\theta) d\theta^1 ... d\theta^n d^m x. \tag{212}$$

To ensure real expectation values for observables, $\rho(.,.)$ must be even(odd) for n even(odd). The condition (210) implies that

$$\rho(x,\theta) = \rho_0(x)\theta^n ... \theta^1 + \text{terms of lower order in } \theta \tag{213}$$

where ρ_0 is a probability density on R^m.

The inequality (211) implies inequalities involving the coefficient functions on the right in Eq.(213). They eventually determine a convex domain $\mathcal{D}$ in a real vector space. Pure states correspond to points on the boundary of $\mathcal{D}$ (which is generally not a manifold).

The CC condition is, unfortunately not generally satisfied by the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$ in super-classical mechanics. To show this, it is adequate to give an example [71]. Taking $\mathcal{A} = C^\infty(R^{0|3}) \equiv \mathcal{G}_3$, we have

$$\rho(\theta) = \theta^3\theta^2\theta^1 + c_\alpha\theta^\alpha. \tag{214}$$

The inequality (211) with $f = a\theta^1 + b\theta^2$ (with a and b arbitrary complex numbers) implies $c_3 = 0$; similarly, $c_1 = c_2 = 0$, giving, finally

$$\rho(\theta) = \theta^3\theta^2\theta^1. \tag{215}$$

There is only one possible state which must be pure. This state does not distinguish, for example, observables $f = a + b\theta^1\theta^2$ with the same 'a' but different 'b', thus verifying the assertion made above.

The fermionic extension of classical mechanics, therefore, appears to have a fundamental inadequacy [at least when the states are restricted as in Eq.(212)]; no wonder, therefore, that it is not realized by systems in nature.

The argument presented above, however, does not apply to the $n = \infty$ case.

H. Quantum-Classical Correspondence

In this subsection, it will be shown that supmech permits a transparent treatment of quantum-classical correspondence satisfying the criteria laid down in the introduction. In contrast to the general practice in this domain, we shall, in our treatment, be careful about the domains of operators and avoid some usual pitfalls in the treatment of the $\hbar \to 0$ limit.

Our strategy will be to start with a quantum Hamiltonian system, transform it to an isomorphic supmech Hamiltonian system involving phase space functions and $\star$-products (Weyl-Wigner-Moyal formalism [116-118]) and show that, in this latter Hamiltonian system, the subclass of phase space functions in the system algebra which go over to smooth functions in the $\hbar \to 0$ limit yield the corresponding classical Hamiltonian system. For simplicity, we restrict ourselves to the case of a spinless nonrelativistic particle though the results obtained admit trivial generalization to systems with phase space R^{2n}.

In the existing literature, the works on quantum-classical correspondence closest to the present treatment are those of Ref.[119-121]; some results from these works, especially Liu[119]

are used below. Ref.[122] is a comprehensive work reporting on some detailed features of quantum-classical correspondence employing some techniques of noncommutative geometry; its theme, however, is very different from ours.

In the case at hand, we have the quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H} = L^2(R^3), \mathcal{D} = \mathcal{S}(R^3)$ and $\mathcal{A}$ is the algebra of the spinless Galilean particle treated in section VII D as a standard quantum system. As in Eq.(190), we shall take the potential function V to be a function of $\mathbf{X}$ only. For $A \in \mathcal{A}$ and ϕ, ψ normalized elements in $\mathcal{D}$, we have the well defined quantity

$$(\phi, A\psi) = \int \int \phi^*(y) K_A(y, y') \psi(y') dy dy' \tag{216}$$

where the kernel K_A is a (tempered) distribution. Recalling the definition of Wigner function [117,123] corresponding to the wave function ψ :

$$W_\psi(x, p) = \int_{R^3} exp[-ip.y/\hbar] \psi(x + \frac{y}{2}) \psi^*(x - \frac{y}{2}) dy \tag{217}$$

and defining the quantity $A_W(x, p)$ by

$$A_W(x, p) = \int exp[-ip.y/\hbar] K_A(x + \frac{y}{2}, x - \frac{y}{2}) dy \tag{218}$$

(note that W_ψ is nothing but the quantity P_W where P is the projection operator $|\psi >< \psi|$ corresponding to ψ) we have

$$(\psi, A\psi) = \int \int A_W(x, p) W_\psi(x, p) dx dp. \tag{219}$$

Whereas the kernels K_A are distributions, the objects A_W are well defined functions. For example,

$$\begin{aligned} A = I : \quad & K_A(y, y') = \delta(y - y') \quad A_W(x, p) = 1 \\ A = X_j : \quad & K_A(y, y') = y_j \delta(y - y') \quad A_W(x, p) = x_j \\ A = P_j : \quad & K_A(y, y') = -i\hbar \frac{\partial}{\partial y_j} \delta(y - y') \quad A_W(x, p) = p_j. \end{aligned}$$

The Wigner functions W_ψ are generally well-behaved functions. We shall use Eq.(219) to characterize the class of functions A_W and call them Wigner-Schwartz integrable (WSI) functions [i.e. functions integrable with respect to the Wigner functions corresponding to the Schwartz functions in the sense of Eq.(217)]. For the relation of this class to an appropriate class of symbols in the theory of pseudodifferential operators, we refer to Wong [123] and references therein.

The operator A can be reconstructed (as an element of $\mathcal{A}$) from the function A_W :

$$\begin{aligned} &(\phi, A\psi) = \\ &(2\pi\hbar)^{-3} \int \int \int exp[ip.(x - y)/\hbar] \phi^*(x) A_W(\frac{x + y}{2}, p) \psi(y) dp dx dy. \end{aligned} \tag{220}$$

Replacing, on the right hand side of Eq.(217), the quantity $\psi(x+\frac{y}{2})\psi^*(x-\frac{y}{2})$ by $K_\rho(x+\frac{y}{2}, x-\frac{y}{2})$ where $K_\rho(.,.)$ is the kernel of the density operator ρ, we obtain the Wigner function $\rho_W(x,p)$ corresponding to ρ. Eq.(219) then goes over to the more general equation

$$Tr(A\rho) = \int\int A_W(x,p)\rho_W(x,p)dxdp. \tag{221}$$

The Wigner function ρ_W is real but generally not non-negative.

Introducing, in R^6, the notations ξ = (x,p), $d\xi = dxdp$ and $\sigma(\xi,\xi^{'}) = p.x^{'} - x.p^{'}$ (the symplectic form in R^6), we have, for A,B $\in \mathcal{A}$

$$\begin{aligned}(AB)_W(\xi) &= (2\pi)^{-6}\int\int exp[-i\sigma(\xi-\eta,\tau)]A_W(\eta+\frac{\hbar\tau}{4}). \\ &\quad .B_W(\eta-\frac{\hbar\tau}{4})d\eta d\tau \\ &\equiv (A_W \star B_W)(\xi). \end{aligned} \tag{222}$$

The product $\star$ of Eq.(222) is the *twisted product* of Liu [119] and the $\star$- *product* of Bayen et al [124]. The associativity condition $A(BC) = (AB)C$ implies the corresponding condition $A_W \star (B_W \star C_W) = (A_W \star B_W) \star C_W$ in the space $\mathcal{A}_W$ of WSI functions which is a complex associative non-commutative, unital *-algebra (with the star-product as product and complex conjugation as involution). There is an isomorphism between the two star-algebras $\mathcal{A}$ and $\mathcal{A}_W$ as can be verified from equations (220) and (218).

Recalling that, in the quantum Hamiltonian system $(\mathcal{A}, \omega_Q, H)$ the form ω_Q is fixed by the algebraic structure of $\mathcal{A}$ and noting that, for the Hamiltonian H of Eq.(111)[with V = V($\mathbf{X}$)],

$$H_W(x,p) = \frac{p^2}{2m} + V(x), \tag{223}$$

we have an isomorphism between the supmech Hamiltonian systems $(\mathcal{A}, \omega_Q, H)$ and $(\mathcal{A}_W, \omega_W, H_W)$ where $\omega_W = -i\hbar\omega_c$. Under this isomorphism, the quantum mechanical PB (160) is mapped to the Moyal bracket [119]

$$\{A_W, B_W\}_M \equiv (-i\hbar)^{-1}(A_W \star B_W - B_W \star A_W). \tag{224}$$

For functions f,g in $\mathcal{A}_W$ which are smooth and such that $f(\xi)$ and $g(\xi)$ have no $\hbar-$dependence, we have, from Eq.(222),

$$f \star g = fg - (i\hbar/2)\{f,g\}_{cl} + O(\hbar^2). \tag{225}$$

The functions $A_W(\xi)$ will have, in general, some $\hbar$ dependence and the $\hbar \to 0$ limit may be singular for some of them [125]. We denote by $(\mathcal{A}_W)_{reg}$ the subclass of functions in $\mathcal{A}_W$ whose $\hbar \to 0$ limits exist and are smooth (i. e. C^∞) functions; moreover, we demand that the Moyal bracket of every pair of functions in this subclass also have smooth limits. This class is easily seen to be a subalgebra of $\mathcal{A}_W$ closed under Moyal brackets. Now, given two functions A_W and B_W in this class, if $A_W \to A_{cl}$ and $B_W \to B_{cl}$ as $\hbar \to 0$ then $A_W \star B_W \to A_{cl}B_{cl}$;

the subalgebra $(\mathcal{A}_W)_{reg}$, therefore, goes over, in the $\hbar \to 0$ limit , to a subalgebra $\mathcal{A}_{cl}$ of the commutative algebra $C^\infty(R^6)$ (with pointwise product as multiplication). The Moyal bracket of Eq.(224) goes over to the classical PB $\{A_{cl}, B_{cl}\}_{cl}$; the subalgebra $\mathcal{A}_{cl}$, therefore, is closed under the classical Poisson brackets. The classical PB $\{,\}_{cl}$ determines the nondegenerate classical symplectic form ω_{cl}. When $H_W \in (\mathcal{A}_W)_{reg}$[which is the case for the H_W of Eq.(223)], the subsystem $(\mathcal{A}_W, \omega_W, H_W)_{reg}$ goes over to the supmech Hamiltonian system $(\mathcal{A}_{cl}, \omega_{cl}, H_{cl})$.

When the $\hbar \to 0$ limits of A_W and ρ_W on the right hand side of Eq.(221) exist (call them A_{cl} and ρ_{cl}), we have

$$Tr(A\rho) \to \int\int A_{cl}(x,p)\rho_{cl}(x,p)dxdp. \tag{226}$$

The quantity ρ_{cl} must be non-negative (and, therefore, a genuine density function). To see this, note that the $\hbar \to 0$ limit preserves products and conjugation and, therefore, maps non-negative operators to non-negative functions. Now if, in Eq.(226), A is a non-negative operator, the left hand side is non-negative for an arbitrarily small value of $\hbar$ and, therefore, the limiting value on the right hand side must also be non-negative. This will prove the non-negativity of ρ_{cl} if the objects A_{cl} in Eq.(226) realizable as classical limits constitute a dense set of non-negative functions in $C^\infty(M)$. This class is easily seen to include non-negative polynomials; good enough.

In situations where the $\hbar \to 0$ limit of the time derivative equals the time derivative of the classical limit [i.e. we have $A(t) \to A_{cl}(t)$ and $\frac{dA(t)}{dt} \to \frac{dA_{cl}(t)}{dt}$], the Heisenberg equation of motion for A(t) goes over to the classical Hamilton's equation for $A_{cl}(t)$. With a similar proviso, one obtains the classical Liouville equation for ρ_{cl} as the classical limit of the von Neumann equation.

Before closing this section, we briefly discuss an interesting point :

For commutative algebras, the inner derivations vanish and one can have only outer derivations. Classical mechanics employs a subclass of such algebras (those of functions on manifolds). It is an interesting contrast to note that, while the standard quantum systems have system algebras with only inner derivations, classical system algebras have only outer derivations. The deeper significance of this is related to the fact that the noncommutativity of standard quantum algebras is tied to the nonvanishing of the Planck constant $\hbar$. [This is seen most transparently in the star product of Eq.(222) above.] In the limit $\hbar \to 0$, the algebra becomes commutative (the star product of functions reduces to ordinary product)and the inner derivations become outer derivations (commutators go over to classical Poisson brackets implying that an inner derivation D_A goes over to the Hamiltonian vector field $X_{A_{cl}}$).

VIII. MEASUREMENTS IN QUANTUM MECHANICS

In this section we shall employ the formalism of section VI B to the treatment of measurements in QM taking both the measured system and the apparatus to be quantum systems. We shall, however, not adopt the von Neumann procedure [42,19] of introducing vector states for

the pointer positions (which is the basic cause of all problems in the quantum measurement theory). Instead we shall assign density operators to the pointer states and exploit the fact that the apparatus admits a classical description to a very good approximation. We shall do this by using the phase space description of the QM of the apparatus (the Weyl-Wigner-Moyal formalism) and then go to the classical approximation (exploiting the fact that supmech accommodates both classical and quantum mechanics as special subdisciplines). The undesirable macroscopic superpositions (of system + apparatus pure states), whose presence in the von Neumann type treatments constitutes the measurement problem, are shown to be suppressed when observations on the apparatus are restricted to macroscopically distinguishable pointer readings.

We shall start by putting the measurement problem in proper perspective.

A. The measurement problem in quantum mchanics

A measurement is an activity in which a system [about which some information is desired — to be called the 'measured system' (denoted here as S); it may be microscopic or macroscopic], prepared in a specified state, is made to interact with a (generally macroscopic) system called the 'apparatus' (denoted here as A) so as to eventually produce a phenomenon accessible through sensory perception (typically a pointer reading) or a permanent record (which may be noted at convenience). The pointer reading or record (the 'measurement outcome') is a numerical value which is interpreted (by employing an underlying theory and some common sense logic formalizable in terms of classical physics) as the value of some physical quantity (an observable). Thus one can talk about measurement of an observable of a system (when the system is in a given state).

In quantum theoretic treatments, a *value* of an observable (a self-adjoint operator) is understood to be a real number in its spectrum [24, p.115]. Supmech events of the type $\nu(E)$ of section IV A can be associated with domains in the spectrum of an observable A by employing the resolution of identity (172) corresponding to A [so that $\Omega = \sigma(A)$] :

$$\nu(\Delta) = \int_{\Delta} d\mu(\lambda) |\lambda><\lambda| \tag{227}$$

where Δ is a measurable subset of $\Omega = \sigma(A)$. These $\nu(\Delta)$s should, more appropriately, be called *quantum events*. We have already used objects of the form (227) [see Eq.(184)]. Given a state ϕ in which the system S is prepared, we have, recalling Eq.(56),

$$p_{\phi}(\Delta) = \phi(\nu(\Delta)) \tag{228}$$

as the probability that, on measurement of the observable A, its value will be found in the domain Δ. These probabilities are the predictions of the underlying theory. Verification of the theory consists in comparing these probabilities with the appropriate relative frequencies in repeated measurements.

We consider, for simplicity, the measurement of an observable (of a quantum system S) represented by a self-adjoint operator F (acting in an appropriate domain in the Hilbert space

$\mathcal{H}_S$ of S) having a nondegenerate spectrum with the eigenvalue equations $F|\psi_j>=\lambda_j|\psi_j>$ (j = 1,2,...). The apparatus A is chosen such that, to each of the eigenvalues λ_j corresponds a pointer position M_j. If the system is initially in an eigenstate $|\psi_j>$, the apparatus is supposedly designed to give, after the measurement interaction, the pointer reading M_j; the outcome of the measurement is then understood as λ_j. A question immediately arises : 'What is the measurement outcome when the initial state of the system S is a superposition state $|\psi>=\sum_j c_j|\psi_j>$?' The theory is expected to provide a clear answer. To find the answer, we must consider the dynamics of the coupled system (S + A) with an appropriate measurement interaction.

The standard treatment of measurements in QM [42,19,46,24,126] is due to von Neumann who emphasized that quantum mechanics being, supposedly, a universally applicable theory, every system is basically quantum mechanical; to have a consistent theory of measurement, we must, therefore, treat the apparatus A also quantum mechanically. Accordingly, one introduces a Hilbert space $\mathcal{H}_A$ for the apparatus A; the pointer positions M_j are assumed to be represented by the state vectors $|\mu_j>$ in this space. The Hilbert space for the coupled system (S + A) is taken to be $\mathcal{H}=\mathcal{H}_S\otimes\mathcal{H}_A$.

The measurement interaction is elegantly described [24,126] by a unitary operator U on $\mathcal{H}$ which, acting on the initial state of (S+A) (with the system S in the initial state in which it is prepared for the experiment and the apparatus in the 'ready' state which we denote as $|\mu_0>$) gives an appropriate final state. We shall assume the measurement to be *ideal* which is supposedly such that [127] 'when the measured system is initially in an eigenstate of the measured observable, the measurement leaves it in the same state.' In this case, the measurement outcome must be the corresponding eigenvalue which must be indicated by the final pointer position. This implies

$$U(|\psi_j>\otimes|\mu_0>)=|\psi_j>\otimes|\mu_j>\,. \tag{229}$$

For S in the initial state $|\psi>=\sum c_j|\psi_j>$, the final (S + A)- state must be, by linearity of U,

$$|\Psi_f>\equiv U[(\sum_j c_j|\psi_j>)\otimes|\mu_0>]=\sum_j c_j[|\psi_j>\otimes|\mu_j>]. \tag{230}$$

Note that the right hand side of Eq.(218) is a superposition of the quantum states of the (generally *macroscopic*) system (S + A).

Experimentally, however, one does not observe such superpositions. Instead, one obtains, in each measurement, a definite outcome λ_j corresponding to the final (S + A)-state $|\psi_j>\otimes|\mu_j>$. To account for this, von Neumann postulated that, after the operation of the measurement interaction as above, a discontinuous, noncausal and instantaneous process takes place which changes the state $|\Psi_f>$ to the state represented by the density operator

$$\rho_f = \sum_i \tilde{P}_i|\Psi_f><\Psi_f|\tilde{P}_i \tag{231}$$

$$= \sum_j |c_j|^2[|\psi_j><\psi_j|\otimes|\mu_j><\mu_j|] \tag{232}$$

with $\tilde{P}_i = |\psi_i >< \psi_i| \otimes I_A$ where I_A is the identity operator on $\mathcal{H}_A$. This is referred to as von Neumann's *projection postulate* and the phenomenon with the above process as the underlying process the *state vector reduction* or *wave function collapse*. Eq.(232) represents, in the von Neumann scheme, the (S +A)-state on the completion of the measurement. It represents an ensemble of (S + A)-systems in which a fraction $p_j = |c_j|^2$ appears in the j the product state in the summand. With the projection postulate incorporated, the von Neumann formalism, therefore, predicts that, in a measurement with the system S initially in the superposition state as above,

(i) the measured values of the observable F are the random numbers λ_j with respective probabilities $|c_j|^2$;

(ii) when the measurement outcome is λ_j, the final state of the system is $|\psi_j >$.

Both the predictions are in complete accord with experiment.

The main problem with the treatment of a quantum measurement given above is the ad-hoc nature of the reduction postulate. Moreover, having to invoke a discontinuous, acausal and instantaneous process is a very unpleasant feature of the formalism. The so-called measurement problem in QM is essentially the problem of explaining the final state (232) without introducing anything ad-hoc in the theoretical treatment. This means that one should either give a convincing dynamical explanation of the reduction process or else circumvent it.

A serious attempt to solve this problem within the framework of traditional QM invokes the interaction of the system (S + A) with the environment which results in a rapid suppression of the interference terms in the quantity $\zeta \equiv |\Psi_f >< \Psi_f| - \rho_f$ (environment-induced decoherence [25]). A critical evaluation of this approach, however, shows [128,129,126] that it does not really solve the problem. In the decoherence formalism, the reduced density opertor of (S + A) is obtained by taking trace (over the environment $\mathcal{E}$) of the density operator of the system(S + A + $\mathcal{E}$). Zurek [130] interprets this as ignoring the uncontrolled and unmeasured degrees of freedom. This is supposed to be taken as similar to the procedure of deriving the probability 1/2 for 'heads' as well as 'tails' in the experiment of tossing a fair coin by averaging over the uncontrolled and unmeasured degrees of freedom of the environment of the coin.

The two procedures are, however, substantially different [128]. In the coin toss experiment, when, ignoring the environment, we claim that the probability of getting 'heads' in a particular toss of the coin is 1/2, we can also claim that we do, in fact, get *either* 'heads' *or* 'tails' on each particular toss. A definite outcome can be predicted if we take into consideration appropriate environmental parameters and details of initial conditions of the throw. In the case of a quantum measurement (as treated in the decoherence formalism), however, we cannot claim that, taking the environment into consideration, a definite outcome of the experiment will be predicted. In fact, taking the environment into account will give us back a troublesome equation of the form of Eq.(230) [with 'A' replaced by 'A + $\mathcal{E}$'] which is obtained in a von Neumann type treatment of the system (S + A + $\mathcal{E}$).

In the (relative state)/(many worlds) interpretation [131] of QM, one takes the view that Eq.(230) *is* the final outcome of the measurement. In this view, this equation is to be interpreted

as a splitting of the state vector of (S + A) into various branches (often called the *Everett branches*) only one of which we observe. This approach is very uneconomical and intuitively unappealing. Moreover, the so-called preferred basis problem [25,126] which arises in the von Neumann treatment remains unsolved in this approach. For a discussion of some recent attempts to justify this approach, see [132] and references therein.

An approach which offers an appealing solution of the measurement problem is Bohmian mechanics [133,22,134,135]. In this approach, the complete specification of state of a system involves, besides the wave function $\psi(q,t)$ (which is promoted to the status of a physical field associated with the system), the functions $q_\alpha(t)$ describing the configuration space trajectory of the system. The wave function $\psi(q,t)$ serves as a guidance field for the (highly non-Newtonian) motion of the system point in the configuration space [essentially analogous to the way the Hamilton - Jacobi function S(q,t) serves, in classical mechanics, as a guidance field for the motion of the system point in configuration space]. The functions $q_\alpha(t)$ serve as 'hidden variables'. They serve to pick up unique outcomes in measurement situations. [At any time t, q(t) has a definite value. At the end of a measurement, the system trajectory is expected to be in only one of the various configuration space domains corresponding to the different outcomes in the superposition (230).] Born rule probabilities emerge for the observer who cannot access the additional information contained in q(t).

This approach, however, has problems of its own. It has serious problems in the relativistic domain and quantum field theory. Moreover, the physics of the ψ field has some unappealing/unconvincing aspects (which relate to its supposedly being a *physical* field; for example, it influences particle motion, but is not influenced by it).

Besides the above mentioned attempts, there are others, notably, the dynamical collapse models [136-142] which involve significant modifications of quantum mechanics. The unitary evolution of traditional QM is replaced by a stochastic unitary one :

$$d\psi(t) = (Adt + BdW_t)\psi(t)$$

where W_t is a Wiener process and A and B are suitably chosen operators. Heuristically, the idea is that, quantum mechanics may be modified by a low level universal noise, akin to Brownian motion (possibly arising from physics at the Planck scale) which, in certain situations, causes reduction of the state vector. This approach has some successes but cannot be claimed to have successfully replaced the traditional QM.

The problem really lies with the assignment of pure states to the pointer positions in the quantum theoretic treatment of the apparatus in the von Neumann approach. An apparatus is a quantum system admitting a classical description to a very good approximation. A straightforward quantum theoretic treatment should employ density operators corresponding to the various pointer positions which, in the phase space description (in the Weyl-Wigner-Moyal formalism), are approximated well by appropriate classical phase space densities. Pure states correspond to a special subclass of density operators (those of the form $|\psi><\psi|$) and one should give justification for employing them; no such justification is given in the traditional

von Neumann treatment.

We shall see below that a straightforward treatment along the above-mentioned lines is adequate. The von Neumann reduction will be seen as the prescription for taking into consideration the fact that observations relating to the apparatus are restricted to observing the *macroscopically distinguishable* pointer positions; the collapsed final (S+A)-state will be seen to emerge as the effective final state of an observationally constrained system.

B. Supmech treatment of a quantum measurement

We shall now treat the (S +A) system in the supmech framework of section VI B treating both, the system S and the apparatus A, as quantum Hamiltonian systems. Given the two quantum triples $(\mathcal{H}_S, \mathcal{D}_S, \mathcal{A}_S)$ and $(\mathcal{H}_A, \mathcal{D}_A, \mathcal{A}_A)$ corresponding to S and A, the quantum triple corresponding to (S+A) is $(\mathcal{H}_S \otimes \mathcal{H}_A, \mathcal{D}_S \otimes \mathcal{D}_A, \mathcal{A}_S \otimes \mathcal{A}_A)$.

Two important points about the apparatus are :
(i) the observations relating to it are restricted to the pointer positions M_j;
(ii) different pointer positions are macroscopically distinguishable.

These points will play a crucial role in our treatment below.

A general pointer observable for A is of the form

$$J = \sum_j b_j P_j \tag{233}$$

where P_j is the projection operator onto the space of states in $\mathcal{H}_A$ corresponding to the pointer position M_j (considered as an apparatus property; for a detailed treatment of the relationship between classical properties and quantum mechanical projectors, see Omnes[24,128] and references therein) and b_js are real numbers such that $b_j \neq b_k$ for $j \neq k$. The phase space function P_j^W corresponding to the projector P_j is supposedly approximated well by a function P_j^{cl} on the phase space Γ of the apparatus A (the $\hbar \to 0$ limit of P_j^W). Now, in Γ, there must be non-overlapping domains D_j corresponding to the pointer positions M_j. In view of the point (i) above, different points in a single domain D_j are not distinguished by the experiment. We can, therefore, take P_j^{cl} to be proportional to the characteristic function χ_{D_j} of the domain D_j; it follows that the phase space function J^W corresponding to the operator J above is approximated well by the classical pointer observable

$$J^{cl} = \sum_j b'_j \chi_{D_j} \tag{234}$$

where b'_js have properties similar to the b_js above.

The pointer states $\phi_j^{(A)}$ corresponding to the pointer positions M_j are density operators of the form (constant)P_j; the phase space functions corresponding to these states are approximated well by the classical phase space density functions $\rho_j^{cl} = V(D_j)^{-1}\chi_{D_j}$ where V(D) is the phase space volume of the domain D. [Note. If $V(D_j)$ is infinite, one can treat ρ_j^{cl} as a function on Γ which vanishes outside D_j and varies very slowly in D_j.]

We shall take $H_{int} = F \otimes K$ (absorbing the coupling constant in K) where F is the measured quantum observable and K is a suitably chosen apparatus variable J. We shall make the usual assumption that, during the measurement interaction, H_{int} is the dominant part of the total Hamiltonian ($H \simeq H_{int}$). The unitary operator U of subsection A describing the measurement interaction in the von Neumann scheme must now be replaced by the measurement operator in supmech [which implements the appropriate canonical transformation on the states of the (S +A) system] given by $M \equiv exp[\tau \tilde{\partial}_H]$ where $\tau = t_f - t_i$ is the time interval of measurement interaction and $\tilde{\partial}_H$ is the evolution generator in the supmech Liouville equation (65) [it is the transpose of the operator ∂_H, the evolution generator in Eq.(64)].

Assuming, again, that the measurement is ideal and denoting the 'ready state' of the apparatus by $\phi_0^{(A)}$, we have the following analogue of Eq.(229):

$$M(|\psi_j >< \psi_j| \times \phi_0^{(A)}) = |\psi_j >< \psi_j| \times \phi_j^{(A)}. \tag{235}$$

When the system is initially in the superposition state $|\psi >$ as above, the initial and final (S+A)- states are

$$\Phi_{in} = |\psi >< \psi| \times \phi_0^{(A)}; \quad \Phi_f = M(\Phi_{in}). \tag{236}$$

Note that the 'ready' state may or may not correspond to one of the pointer readings. (In a voltage type measurement,it does; in the Stern-Gerlach experiment with spin half particles, it does not.) For the assignment of the Γ-domain to the 'ready' state, the proper interpretation (which covers both the situations above) of the ready state is 'not being in any of the (other) pointer states'. Accordingly, we assign, to this state, the domain

$$\tilde{D}_0 \equiv \Gamma - \cup_{j \neq 0} D_j \tag{237}$$

where the condition $j \neq 0$ on the right is to be ignored when the 'ready' state is not a pointer state.

The final state expected on applying the von Neumann reduction is [the analogue of the state (232) above]

$$\begin{aligned} \Phi'_f &= \sum_j |c_j|^2 [|\psi_j >< \psi_j| \times \phi_j^{(A)}] \\ &= M\left(\sum_j |c_j|^2 [|\psi_j >< \psi_j| \times \phi_0^{(A)}] \right) \end{aligned} \tag{238}$$

where we have used the fact that, in supmech, a canonical transformation on states preserves convex combinations.

Recalling the apparatus feature (i) above, we shall now show Eq.(238) to be the effective final states of observationally constrained system (S+A). We shall show that, for a general system observable A and a pointer observable J, the expectation values of the observable $A \otimes J$

in the states Φ_f and Φ'_f are equal (to a very good approximation). We, therefore, look for the vanishing of the quantity

$$W \equiv (\Phi_f - \Phi'_f)(A \otimes J) = M(R)(A \otimes J)$$

where

$$R = [\sum_{j \neq k} c_k^* c_j |\psi_j >< \psi_k|] \times \phi_0^{(A)}.$$

[Note that R is not an (S +A)-state; here M has been implicitly extended by linearity to the dual space of the algebra $\mathcal{A}_S \otimes \mathcal{A}_A$.] Transposing the M operation to the observables, we have

$$\begin{aligned} W &= R[exp(\tau \partial_H)(A \otimes J)] \\ &= \int_\Gamma d\Gamma \rho_0^{(A)W} \sum_{j \neq k} c_k^* c_j < \psi_k | exp(\tau \partial_{H'})(A \otimes J^W) | \psi_j > \end{aligned} \tag{239}$$

where we have adopted the phase space description of the QM of the apparatus, $d\Gamma$ is the phase space volume element, $\rho_0^{(A)W}$ is the Wigner function corresponding to the state $\phi_0^{(A)}$ and $H' = F \otimes K^W$. (The subscript W of section VII H has been replaced by a superscript here.) Using equations (154) and (224), we have

$$\begin{aligned} \partial_{H'}(A \otimes J^W) &= \{F \otimes K^W, A \otimes J^W\} \\ &= (-i\hbar)^{-1} \left([F, A] \otimes \frac{K^W * J^W + J^W * K^W}{2} \right. \\ &\quad \left. + \frac{FA + AF}{2} \otimes (K^W * J^W - J^W * K^W) \right). \end{aligned} \tag{240}$$

Given the fact that the apparatus is well described classically, we have $K^W \simeq K^{cl}$ and $J^W \simeq J^{cl}$ to a very good approximation. This gives

$$\partial_{H'}(A \otimes J^W) \simeq (-i\hbar) K^{cl} J^{cl} [F, A]$$

which, in turn, implies

$$< \psi_k | exp(\tau \partial H')(A \otimes J^W) | \psi_j > \simeq exp[\frac{i}{\hbar}(\lambda_k - \lambda_j) K^{cl} \tau] J^{cl} < \psi_k | A | \psi_j >$$

We now have, replacing $\rho_0^{(A)W}$ by its classical approximation $\rho_0^{(A)cl}$,

$$W \simeq \int_{\tilde{D}_0} d\Gamma \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j exp[\frac{i}{\hbar}(\lambda_k - \lambda_j) K^{cl} \tau] J^{cl} < \psi_k | A | \psi_j > . \tag{241}$$

Let

$$< K^{cl} >_0 \equiv \int_{\tilde{D}_0} K^{cl} \rho^{(A)cl} d\Gamma \tag{242}$$

(the mean value of K^{cl} in the domain $\tilde{D}_0$) Putting $K^{cl} =< K^{cl} >_0 s$, taking s to be one of the integration variables and writing $d\Gamma = dsd\Gamma'$, we have

$$W \simeq \int_{\tilde{D}_0} dsd\Gamma' \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j exp[\frac{i}{\hbar}\eta_{jk}s]J^{cl} < \psi_k|A|\psi_j > \tag{243}$$

where

$$\eta_{jk} = (\lambda_k - \lambda_j) < K^{cl} >_0 \tau. \tag{244}$$

Note that s is a real dimensionless variable with domain of integration of order unit length.

We shall now argue that, for $j \neq k$,

$$|\eta_{jk}| >> \hbar. \tag{245}$$

(This is not obvious; when F is a component of spin, for example, the λs are scalar multiples of $\hbar$.) To this end, we invoke the apparatus feature (ii) above. A fairly straightforward way of formulating a criterion for macroscopic distinguishability of different pointer positions would be to identify a quantity of the dimension of action which could be taken to characterize the physical separation between two different pointer positions and show that its magnitude is much larger than $\hbar$. The objects η_{jk} (for $j \neq k$)are quantities of this type. The inequality (244) then follows from the assumed macroscopic distinguishability of different pointer positions. Another, essentially equivalent, way of seeing this is to treat Eq.(244) as the time-energy uncertainty inequality $|\Delta E \Delta t| >> \hbar$ where $\Delta t = \tau$ and ΔE is the difference between the energy values corresponding to the apparatus locations in two different domains D_j and D_k in Γ. Recalling that $H \simeq H_{int}$ during the relevant time interval, we have $\Delta E \simeq (\lambda_k - \lambda_j) < K^{cl} >_0$.

The large fluctuations implied by Eq.(244) wipe out the integral above giving $W \simeq 0$ as desired. This completes the derivation of the von Neumann reduction rule. The derivation makes it clear as to the sense this reduction rule should be understood : it is a prescription for obtaining the effective final state of the observationally constrained (S + A) system.

Eq.(243), followed by the reasoning above, represents, in a *live* form, the operation of environment-induced decoherence. To see this, note that, the domain $\tilde{D}_0$ may be taken to represent the internal environment [127] of the apparatus. With this understanding, the mechanism wiping out the unwanted quantum interference terms is, indeed, the environment-induced decoherence. In the treatment presented here (in which the apparatus is 'respectfully' treated as a *system*), this mechanism becomes automatically operative. (Even the external environment can be trivially included by merely saying that the system A above represents 'the apparatus and its environment'.)

In the next subsection, we shall illustrate the operation of the formalism developed above in a concrete situation.

C. Example : the Stern-Gerlach experiment

As an illustration, we consider the Stern-Gerlach experiment [21,24,143] with, say, silver atoms (which means spin s = $\frac{1}{2}$). A collimated beam of (unpolarized) silver atoms is made to pass through inhomogeneous magnetic field after which the beam splits into two beams corresponding to atoms with $S_z = \pm\frac{\hbar}{2}$. The spin and magnetic moment operators of an atom are $\mathbf{S} = \frac{\hbar}{2}\sigma$ and $\mu = g\mathbf{S}$ (where g is the magnetogyric ratio). Let the magnetic field be $\mathbf{B}(\mathbf{r}) = B(z)\mathbf{e_3}$ (in obvious notation). (Refinements [144] introduced to ensure the condition $\bigtriangledown.\mathbf{B} = 0$ do not affect the essential results obtained below.) We have

$$H_{int} = -\mu.\mathbf{B} = -gB(z)S_3. \tag{246}$$

The force on an atom, according to Ehrenfest's theorem, is

$$\mathbf{F} = -\bigtriangledown < -\mu.\mathbf{B} >= g\frac{dB(z)}{dz} < S_3 > \mathbf{e_3} \tag{247}$$

where the average is taken in the quantum state of the atom. During the experiment, the internal state of the atom remains unchanged (to a very good approximation); only its center of mass $\mathbf{r}$ and spin $\mathbf{S}$ have significant dynamics. In this experiment, S_3 is the measured quantum observable and $\mathbf{r}$ acts as the operative apparatus variable.

Let us assume that the beam initially moves in the positive x-direction, the pole pieces are located in the region $x_1 \leq x \leq x_2$ and the detectors located in the plane $x = x_3 > x_2$ (one each in the regions $z > 0$ and $z < 0$; these regions contain the emergent beams of silver atoms corresponding, respectively, to $S_3 = +\frac{\hbar}{2}$ and $S_3 = -\frac{\hbar}{2}$). We have, in the notation used above, $F = S_3$ and K = - g B(z). Assuming the experiment to start when the beam reaches at $x = x_1$, the phase space of the apparatus is

$$\Gamma = \{(x, y, z, p_x, p_y, p_z) \in R^6; x \geq x_1, *\} \tag{248}$$

where * indicates the restriction that, for $x_1 \leq x \leq x_2$, the space available for the movement of atoms is the one between the two pole pieces. For the order of magnitude calculation below, we shall ignore the shape of the pole pieces and take * to imply $z_1 \leq z \leq z_2$.

The domains D_1 and D_2 corresponding to the two pointer positions are

$$\begin{aligned} D_1 &= \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z > 0\} \\ D_2 &= \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z < 0\}; \end{aligned}$$

the domain $\tilde{D}_0 = \Gamma - (D_1 \cup D_2)$. For simplicity, let us take $B(z) = b_0 + b_1 z$ where b_0 and b_1 are constants. For $j \neq k$, we have $\lambda_j - \lambda_k = \pm\hbar$. The relevant integral is [see Eq.(243) above]

$$I = \int_{z_1}^{z_2} dz(...)exp[\pm\frac{i}{\hbar}\mu b_1 z\tau] \tag{249}$$

where $\mu = g\hbar$. Putting $z = (z_2 - z_1)u$, the new integration variable u is a dimensionless variable taking values in a domain of length of order one. The quantity of interest is

$$|\eta| = \mu|b_1|(z_2 - z_1)\tau. \tag{250}$$

According to the data in Ref.[143] and Ref.[145, problem 4.6], we have (v_x is the x- component of velocity of the silver atom)

$$|b_1| \sim |\frac{dB}{dz}| \sim 10^5 \ gauss/cm$$
$$z_2 - z_1 \simeq 1mm, \quad v_x \sim 500 \ m/sec$$
$$x_2 - x_1 = 3cm, \quad x_3 - x_2 = 20 \ cm$$

This gives

$$\tau \sim \frac{x_3 - x_1}{v_x} \sim 5 \times 10^{-4} sec.$$

Denoting the Bohr magneton by μ_b and putting $\mu \sim \mu_b \simeq 0.9 \times 10^{-20}$ erg/gauss, we have $|\eta| \sim 10^{-19}$erg-sec. With $\hbar \simeq 1.1 \times 10^{-27}$ erg-sec, we have, finally $(|\eta|/\hbar) \sim 10^8$, confirming the strong suppression of the undesirable quantum interferences.

IX. AXIOMS

We shall now write down a set of axioms covering the work presented in the preceding sections. Before the statement of axioms, a few points are in order :

(i) These axioms are meant to be provisional; the 'final' axioms will, hopefully, be formulated (not necessarily by the present author) after a reasonably satisfactory treatment of quantum theory of fields and space-time geometry in the spirit of the present work has been given.
(ii) The terms 'system', 'observation', 'experiment' and a few other 'commonly used' terms will be assumed to be understood. The term 'relativity scheme' employed below will be understood to have its meaning as explained in section V A.
(iii) The 'universe' will be understood as the largest possible observable system containing every other observable system as a subsystem.
(iv) By an *experimentally accessible system* we shall mean one whose 'identical' (for all practical purposes) copies are reasonably freely available for repeated trials of an experiment. Note that the universe and its 'large' subsystems are not included in this class.
(v) The term 'system' will, henceforth will normally mean an experimentally accessible one. Whenever it is intended to cover the universe and/or its large subsystems (this will be the case in the first two axioms only), the term system* will be used.

The axioms will be labelled as **A1,..., A8**.

A1.(Probabilistic framework; System algebra and states)
(a) *System algebra; Observables.* A system* S has associated with it a supmech-admissible (as defined in section IV A) superalgebra $\mathcal{A} = \mathcal{A}^{(S)}$. (Its elements will be denoted as A,B,...). Observables of S are elements of the subset $\mathcal{O}(\mathcal{A})$ of even Hermitian elements of $\mathcal{A}$.
(b) *States.* States of the system*, also referred to as the states of the system algebra $\mathcal{A}$(denoted by the letters $\phi, \psi, ..$), are defined as (continuous) positive linear functionals on $\mathcal{A}$ which are

normalized [i.e. $\phi(I) = 1$ where I is the unit element of $\mathcal{A}$]. The set of states of $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$ and the subset of pure states by $\mathcal{S}_1(\mathcal{A})$. For any $A \in \mathcal{O}(\mathcal{A})$ and $\phi \in \mathcal{S}(\mathcal{A})$, the quantity $\phi(A)$ is to be interpreted as the expectation value of A when the system is in the state ϕ.
(c) Expectation value of odd elements of $\mathcal{A}$ vanishes in every pure state (hence in every state).
(d) *Compatible completeness of observables and pure states.* The pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$ satisfies the CC condition described in section IV A.
(e) *Experimental situations and probabilities.* An experimental situation has associated with it a positive observable-valued measure (PObVM) as defined in section IV A; it associates, with measurable subset of a measurable space, objects called supmech events which have measure-like properties. Given the system prepared in a state ϕ, the probability of realization of a supmech event $\nu(E)$ is $\phi(\nu(E))$.

A2. *Differential calculus; Symplectic structure.* The system algebra $\mathcal{A}$ of a system* S is such as to permit the development of derivation-based differential calculus based on it; moreover, it is equipped with a real symplectic form ω thus constituting a symplectic superalgebra $(\mathcal{A}, \omega)$ [more generally, a generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega)$ when the derivations are restricted to a distinguished Lie sub-superalgera $\mathcal{X}$ of $SDer(\mathcal{A})$].

A3. *Dynamics.* The dynamics of a system S is described by a one-parameter family of canonical transformations generated by an even Hermitian element H (the Hamiltonian) of $\mathcal{A}$.

The mechanics described by the above-stated axioms will be referred to as supmech. The triple $(\mathcal{A}, \omega, H)$ [or the quadruple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$] will be said to constitute a supmech Hamiltonian system.

A4. *Relativity scheme.* (a) For systems admitting space-time description, the 'principle of relativity' as described in section V A,will be operative.
(b)The admissible relativity schemes will be restricted to (i) Galilean relativity, (ii) special relativity.

A5. *Elementary systems; Material particles.* (a) In each of the admitted relativity schemes, material particles will be understood to be localizable elementary systems (as defined in sections IV G and V A) with positive mass.
(b) The system algebra for a material particle will be (the topological completion of) the one generated by its fundamental observables (as defined in section V A) and the identity element.

A6. *Coupled systems.* Given two systems S_1 and S_2 described as supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \mathcal{S}_1^{(i)}, \omega^{(i)}, H^{(i)})$ (i=1,2), the coupled system $(S_1 + S_2)$ will be described as a supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1, \omega, H)$ with

$$\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}, \quad \mathcal{S}_1 = \mathcal{S}_1(\mathcal{A}) \quad \omega = \tilde{\omega}^{(1)} + \tilde{\omega}^{(2)}$$

[see Eq.(134)] and H as in Eq.(156).

A7. *Quantum systems.* All (experimentally accessible) systems in nature have noncommutative system algebras (and hence are quantum systems); they have a quantum symplectic structure (as defined in section III I) with the universal parameter $b = -i\hbar$.

A8. *Measurements.* In a measurement involving a 'measured system' S and apparatus A
(a) both S and A are standard quantum systems;
(b) the Hamiltonian system $(\mathcal{A}^{(A)}, \mathcal{S}_1^{(A)}, \omega^{(A)}, H^{(A)})$ corresponding to the apparatus admits an equivalent (in the sense of section IV C) phase space realization (in the Weyl-Wigner-Moyal scheme) $(\mathcal{A}_W^{(A)}, \mathcal{S}_{1W}^{(A)}, \omega_W^{(A)}, H_W^{(A)})$;
(c) elements of $\mathcal{A}_W^{(A)}$ and ${\mathcal{S}_1^{(A)}}_W$ appearing in the description of dynamics of the coupled system (S+A) admit $\hbar \to 0$ limits and are approximated well by these limits;
(d) the various pointer positions of the apparatus are macroscopically distinguishable [the macroscopic distinguishability can be interpreted, for example, in terms of an energy-time uncertainty product inequality ($\Delta E \Delta t >> \hbar$) relevant to the experimental situation];
(e) observations on the apparatus are restricted to readings of the output devices (pointers).

Note. Part (b) in this axiom is expected to be redundant and is included 'to be on the safe side'. (A redundancy is excusable if it serves to bring some extra clarity without introducing any inconsistency.)

X. CONCLUDING REMARKS

1. Sometimes the question is raised : 'Why algebras ?' The answer emerging from the present work is : 'Because they provide the right framework for noncommutative symplectic geometry as well as for noncommutative probability and, therefore, are natural objects to employ in the construction of a formalism integrating the two [in the spirit of unification of physics and probability theory envisaged in the formulation of (augmented) Hilbert's sixth problem].

Indeed, as we have seen, for an autonomous development of quantum mechanics, the fundamental objects are algebras and not Hilbert spaces.

2. A contribution of the present work expected to be of some significance to the algebraic schemes in theoretical physics and probability theory is the introduction of the condition of compatible completeness for observables and pure states [axiom A1(d)] which plays an important role in ensuring that the so-called 'standard quantum systems' defined algebraically in section VII A, have faithful Hilbert space-based realizations. It is desirable to formulate necessary and/or sufficient conditions on the superalgebra $\mathcal{A}$ alone (i.e. without reference to states) so that the CC condition is automatically satisfied.

An interesting result, obtained in section VII G, is that the superclassical systems with a finite number of fermionic generators generally do not satisfy the CC condition. This probably explains their non-occurence in nature. It is worth investigating whether the CC condition is related to some stability property of dynamics.

3. Some features of the development of QM in the present work (apart from the fact that it is autonomous) should please theoreticians : there is a fairly broad-based algebraic formalism connected smoothly to the Hilbert space QM; there is a natural place for commutative superselection rules and for the Dirac's bra-ket formalism; the Planck constant is introduced 'by hand' at only one place (at just the right place : the quantum symplectic form) and it appears at all conventional places automatically. Moreover, once the concepts of localization, elementary system and standard quantum system are introduced at appropriate places, it is adequate to define a material particle as a localizable elementary quantum system ; 'everything else' — including the interpretation of the Schrödinger wave function and the Schrödinger equation — is automatic.

4. The treatment of quantum-classical correspondence in section VII H, illustrated with the example of a nonrelativistic spinless particle, makes clear as to how the subject should be treated in the general case : go from the traditional Hilbert space -based description of the quantum system to an equivalent (in the sense of a supmech hamiltonian system) phase space description in the Weyl-Wigner-Moyal formalism, pick up the appropriate subsets in the observables and states having smooth $\hbar \to 0$ limits and verify that the limit gives a commutative supmech Hamiltonian system (which is generally a traditional classical hamiltonian system).

5. The treatment of measurements in QM in section VIII provides a justification for the collapse postulate through straightforward physics : it is the prescription for obtaining the effective final state of an observationally constrained coupled system. The use of pure states for the pointer positions in the von Neumann treatment did not bring out the physics of the system-apparatus correctly. Once that is replaced by the right objects and the other essential ingredient — that observations on the apparatus are restricted to macroscopically distinguishable pointer readings — is appropriately incorporated, things work out quite satisfactorily. The sight of Eq.(243) where one can see the operation of the decohering effect of 'internal environment' of the apparatus in live action, should please theoreticians. The incorporation of the external environment has been reduced to a matter of two lines : just saying that that the symbol 'A' now stands for 'the apparatus and its environment'. It is left to the reader to compare this treatment with that in the scores of papers in literature on 'environment induced decoherence'.

6. The author hopes that the treatment of the supmech formalism in section IV and its successful applications in subsequent sections present it as an object deserving further detailed study (along the lines essentially parallel to the classical symplectic mechanics which is a rich and well developed subject).

7. The title of the present paper was originally intended to be, as announced in Ref.[60,63], 'Supmech: a unified symplectic view of physics'; while preparing the final version of it, however, the author felt that the present title would represent the contents better.

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