Regularity and dimension spectrum
of the equivariant spectral triple
for the odd dimensional quantum spheres

Arupkumar Pal

S. Sundar

Regularity and dimension spectrum of the equivariant spectral triple for the odd dimensional quantum spheres

ARUPKUMAR PAL and S. SUNDAR

November 26, 2008

Abstract

The odd dimensional quantum sphere $S_q^{2\ell+1}$ is a homogeneous space for the quantum $SU(\ell+1)$ group. A generic equivariant spectral triple for $S_q^{2\ell+1}$ on its L_2 space was constructed by Chakraborty & Pal in [4]. We prove regularity for that spectral triple here. We also compute its dimension spectrum and show that it is simple. We give detailed construction of its smooth function algebra and some related algebras that help proving regularity and in the computation of the dimension spectrum. Following the idea of Connes for $SU_q(2)$, we first study another spectral triple for $S_q^{2\ell+1}$ equivariant under torus group action constructed by Chakraborty & Pal in [3]. We then derive the results for the $SU_q(\ell+1)$-equivariant triple in the $q = 0$ case from those for the torus equivariant triple. For the $q \neq 0$ case, we deduce regularity and dimension spectrum from the the $q = 0$ case.

Contents

1

1 Introduction

In noncommutative geometry, the starting point is usually a separable unital C^*-algebra A which is the noncommutative version of a compact Hausdorff space. Associated to this, one has certain invariants like the K-groups and the K-homology groups. In geometry, what one does next is to equip the topological space with a smooth structure so that in particular one can then talk about its de Rham cohomology. In the noncommutative situation, the parallel is to look for an appropriate dense subalgebra of A that will play the role of smooth functions on the space. Given this dense subalgebra, one can compute various cohomology groups associated with it, namely the Hochschild cohomology, cyclic cohomology and the periodic cyclic cohomology, which are noncommutative and far-reaching generalizations of ordinary de Rham homology and cohomology. Question is: how to get hold of this dense subalgebra? One answer to this lies in the notion of a spectral triple, which plays a central role in Connes' formulation of noncommutative geometry.

In ordinary differential geometry, with just a smooth structure on a manifold, one can hardly go very far. In order that one can talk about shapes and sizes of spaces, one needs to bring in extra structure. One example is the Riemannian structure, which gives rise to a Riemannian connection, which in turn enables one to talk about curvature and so on. Other examples of such extra structures are $Spin$ and $Spin^c$ structures. In the presence of these extra structures, one has an operator theoretic data that completely encodes the geometry. In noncommutative geometry, one takes this operator theoretic data as the initial data and this is what goes by the name **spectral triple**.

Definition 1.1 Let $\mathcal{A}$ be an associative unital *-algebra. An **even spectral triple** for $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, D)$ together with a $\mathbb{Z}_2$-grading γ on $\mathcal{H}$ such that

   1. a (complex separable) Hilbert space $\mathcal{H}$,

   2. a *-representation $\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ (usually assumed faithful),

3. a self-adjoint operator D with compact resolvent such that $[D, \pi(a)] \in \mathcal{L}(\mathcal{H})$ for all $a \in \mathcal{A}$,

4. $\pi(a)\gamma = \gamma\pi(a)$ for all $a \in \mathcal{A}$ and $D\gamma = -\gamma D$.

If no grading is present, one calls it an **odd spectral triple**.

The algebra $\mathcal{A}$ appearing in this definition is in general different from the C^*-algebra A one starts with. Typically it is a dense subalgebra in A, big enough so that the K-groups of A and $\mathcal{A}$ coincide.

Since D has compact resolvent, it has finite dimensional kernel. Hence by making a finite rank perturbation, one can make D is invertible. Now if one replaces D with sign D, then one gets the notion of a Fredholm module over the algebra $\mathcal{A}$. If $\mathcal{A}$ is a dense *-subalgebra of a C^*-algebra A, then this Fredholm module extends uniquely and gives a Fredholm module over A. In other words, one gets an element in the K-homology group of A. This gives a map from the K-theory of A to the set of integers via the K-theory–K-homology pairing. Starting from this Fredholm module, one can construct its Chern character, which gives an element in the periodic cyclic cohomology of $\mathcal{A}$, which, in turn, gives a map from the K-theory of A to $\mathbb{Z}$ via the periodic cyclic cohomology–K-theory pairing. The two maps thus obtained are the same. This is the content of the index theorem.

The Chern character is often difficult to compute. And that is where the spectral triple comes into the picture. Under certain hypothesis on the spectral triple, one can construct a cyclic cocycle, i.e. an element in the periodic cyclic cohomology that differs from the Chern character by a coboundary, so that it gives rise to the same map from K-theory to $\mathbb{Z}$. Under mild hypothesis on an invariant known as the dimension spectrum, these cocycles are given in terms of certain residue functionals, that can be relatively easier to compute. This is Connes–Moscovici local index theorem ([5]) which is one of the major results in noncommutative geometry. This was first proved in the context of transverse geometry of foliations, but is much more general in nature and has wider applicability. This was illustrated by Connes in [6], where he made a detailed analysis of the equivariant spectral triple for the quantum $SU(2)$ group constructed in [2]. A similar analysis was later done by Dabrowski et al in [14] for the spectral triple constructed in [8]. Typically, the C^*-algebras associated to quantum groups or their homogeneous spaces are given by a set of generators and relations. While constructing spectral triples, one does it for the associated coordinate function algebra, i.e. the *-subalgebra generated by these generators. This algebra is not closed under the holomorphic function calculus of the C^*-algebra. Therefore one needs to construct the smooth function algebra, prove regularity and compute the dimension spectrum in order to be able to apply the Connes–Moscovici theorem. This was done in [6], where he also defined a symbol map and formulae for computing the residue functionals in terms of the symbol maps were given.

Odd dimensional quantum spheres are higher dimensional analogues of the quantum $SU(2)$. The $(2\ell+1)$ dimensional sphere $S_q^{2\ell+1}$ is a homogeneous space of the quantum group $SU_q(\ell+1)$.

In [4], Chakraborty and Pal constructed a generic spectral triple on the L_2 space of the sphere with nontrivial K-homology class and equivariant under the action of $SU_q(\ell+1)$. The main aim of the present article is to prove that this spectral triple is regular. We also introduce the smooth function algebra and compute the dimension spectrum. The dimension spectrum is shown to be simple so that the Connes–Moscovici local index theorem is applicable to this triple. The local index computation will be taken up in a separate article.

Here is a brief outline of the contents of this paper. In the next section, we recall a few basic notions from [5]. We then collect together a few observations and remarks on tensor products of Fréchet algebras and fix some of the notations. In section 3, we first look at the torus equivariant spectral triple for the spheres and introduce a smooth function algebra, prove regularity and compute the dimension spectrum. The key here is the short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \longrightarrow C(S_q^{2\ell+1}) \longrightarrow C(S_q^{2\ell-1}) \longrightarrow 0,$$

and results by Schweitzer ([12], [13]) on spectral invariance. Using these and using the idea employed by Connes in [6], we build the smooth function algebra over $S_q^{2\ell+1}$ recursively starting from $C^\infty(\mathbb{T})$.

In section 4, we deal with the $SU_q(\ell+1)$-equivariant spectral triple. We first treat the $q=0$ case. Using a decomposition of the L_2-space, we relate it to the torus equivariant triple with a certain multiplicity. Regularity, smooth function algebra and properties of the dimension spectrum all then follow from the results in section 3. Here again the idea is exactly as in [6] for $SU_q(2)$. In subsection 4.4 we take up the $q \neq 0$ case. We take a close look at the representation of the algebra, in particular the images of the generating elements and after a careful analysis, we prove that modulo operators that one can neglect for the purpose of computing the dimension spectrum, things can be deduced from the torus equivariant case again.

2 Preliminaries

2.1 Regular spectral triples

In this subsection we recall some definitions and notions from [5]. Let D be a selfadjoint operator on a Hilbert space $\mathcal{H}$ which is invertible. Define $\mathcal{H}_s = Dom(|D|^s)$ for $s \geq 0$. Then $\mathcal{H}_s$ is a decreasing family of vector subspaces of $\mathcal{H}$. Let $\mathcal{H}_\infty := \cap_{s \geq 0} \mathcal{H}_s$. The subspace $\mathcal{H}_\infty$ is a dense subspace of $\mathcal{H}$.

Definition 2.1 An operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is said to be smoothing if for every $m, n \geq 0$ the operator $|D|^m T |D|^n$ is bounded. The vector space of smoothing operators is denoted by $OP^{-\infty}$.

For $T \in OP^{-\infty}$, define $\|T\|_{m,n} = \||D|^m T |D|^n\|$ for $m, n \geq 0$.

Lemma 2.2 *The vector space $OP^{-\infty}$ is an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ and equipped with the family of seminorms $\|\cdot\|_{m,n}$, it is a Fréchet algebra.*

Let δ be the unbounded derivation $[|D|, \cdot]$. More precisely, $Dom(\delta)$ consists of all bounded operators T which leaves $Dom(|D|)$ invariant and for which $\delta(T) := [|D|, T]$ extends to a bounded operator.

Lemma 2.3 ([7]) *The unbounded derivation δ is a closed derivation i.e. if T_n is a sequence in $Dom(\delta)$ such that $T_n \to T$ and $\delta(T_n) \to S$ then $T \in Dom(\delta)$ and $\delta(T) = S$.*

Define $OP^0 := \{T \in \mathcal{L}(\mathcal{H}) : T \in \cap_n Dom(\delta^n)\}$. The following lemma says that elements of OP^0 are operators on $\mathcal{H}_\infty$.

Lemma 2.4 *Let T be a bounded operator on $\mathcal{H}$. Then the following are equivalent.*

1. *The operator $T \in OP^0$.*

2. *The operator T leaves $\mathcal{H}_\infty$ invariant and $\delta^n(T) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is bounded for every $n \in \mathbb{N}$*

It is easy to see from lemma 2.4 that OP^0 is an algebra and that for every $m \in \mathbb{Z}$ that $|D|^{-m}T|D|^m$ is bounded if $T \in OP^0$. As a consequence it follows that $OP^{-\infty}$ is an ideal in OP^0. Now we recall the notions of regularity and dimension spectrum for a spectral triple.

Definition 2.5 Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We say that $(\mathcal{A}, \mathcal{H}, D)$ is **regular** if $\mathcal{A} + [D, \mathcal{A}] \subset OP^0$.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is $p+$ summable if $|D|^{-p}$ is in the ideal of Dixmier traceable operators $\mathcal{L}^{(1,\infty)}$. In particular, if $(\mathcal{A}, \mathcal{H}, D)$ is $p+$ summable, then $|D|^{-s}$ is trace class for $s > p$. Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple which is $p+$ summable for some p. Let $\mathcal{B}$ be the algebra generated by $\delta^n(\mathcal{A})$ and $\delta^n([D, \mathcal{A}])$. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has **discrete dimension spectrum** $\Sigma \subset \mathbb{C}$ if Σ is discrete and for every $b \in \mathcal{B}$, the function $Trace(b|D|^{-z})$ initially defined for $Re(z) > p$ extends to a meromorphic function with poles only in Σ. We say the dimension spectrum is simple if all the poles are simple.

2.2 Topological tensor products

The C^*-algebras involved in tensor products that we deal with in this paper are all nuclear. Therefore no ambiguities arise due to nonuniqueness of tensor products. Apart from C^*-algebras and their tensor products, we will also deal with Fréchet algebras and their tensor products. Suppose A_1 and A_2 are two Fréchet algebras with topologies coming from the families of seminorms $(\|\cdot\|_\lambda)_{\lambda \in \Lambda}$ and $(\|\cdot\|_{\lambda'})_{\lambda' \in \Lambda'}$. For each pair $(\lambda, \lambda') \in \Lambda \times \Lambda'$, one forms the projective cross norm $\|\cdot\|_{\lambda,\lambda'}$ which is a seminorm on the algebraic tensor product $A_1 \otimes_{alg} A_2$. The family $(\|\cdot\|_{\lambda,\lambda'})_{(\lambda,\lambda') \in \Lambda \times \Lambda'}$ then gives rise to a Fréchet topology on $A_1 \otimes_{alg} A_2$. Completion

with respect to this is the projective tensor product of A_1 and A_2. While talking about tensor product of two Fréchet algebras, we will always mean their projective tensor product and will denote it by $A_1 \otimes A_2$.

We will mainly be concerned with Fréchet algebras sitting inside some $\mathcal{L}(\mathcal{H})$ with Fréchet topology finer than the norm topology. In other words, we will be dealing with Fréchet algebras with faithful continuous representations on Hilbert spaces. Let A_1, A_2 be Fréchet algebras. If $\rho_i : A_i \to \mathcal{L}(\mathcal{H}_i)$ are continuous representations for $i = 0, 1$ where $\mathcal{H}_i$'s are Hilbert spaces, then by the universality of the projective tensor product it follows that there exists a unique continuous representation $\rho_1 \otimes \rho_2 : A_1 \otimes A_2 \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $(\rho_1 \otimes \rho_2)(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2)$. If A_i's are subalgebras of $\mathcal{L}(\mathcal{H}_i)$ then we will call the tensor product representation of $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the natural representation.

Lemma 2.6 *Let $(A_1, \mathcal{H}_1, D_1)$ and $(A_2, \mathcal{H}_2, D_2)$ be regular spectral triples. Assume that the following conditions hold*

1. *The algebras A_1 and A_2 are Fréchet algebras represented faithfully on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*

2. *The selfadjoint operators D_1 and D_2 are positive with compact resolvent.*

3. *For $i = 0, 1$, the unbounded derivations $\delta_i = [D_i, .]$ leave A_i invariant and $\delta_i : A_i \to A_i$ is continuous.*

Let $D := D_1 \otimes 1 + 1 \otimes D_2$. Suppose that the natural representation of $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is faithful. Then the triple $(A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D)$ is a regular spectral triple. More precisely the unbounded derivation $\delta := [D, .]$ leaves the algebra $A_1 \otimes A_2$ invariant and the map $\delta : A_1 \otimes A_2 \to A_1 \otimes A_2$ is continuous.

Proof: Let $\delta' = \delta_1 \otimes 1 + 1 \otimes \delta_2$. Then δ' is a continuous linear operator on $A_1 \otimes A_2$. Clearly $A_1 \otimes_{alg} A_2 \subset Dom(\delta)$ and $\delta = \delta'$ on $A_1 \otimes_{alg} A_2$. Now let $a \in A_1 \otimes A_2$ be given. Choose a sequence $(a_n) \in A_1 \otimes_{alg} A_2$ such that $a_n \to a$ in $A_1 \otimes A_2$. Then $a_n \to a$ in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Since δ' is continuous and because the inclusion $A_1 \otimes A_2 \subset \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is continuous, it follows that the sequence $(\delta'(a_n)) = (\delta(a_n))$ is cauchy in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. As δ is closed, it follows that $a \in Dom(\delta)$ and $\delta(a) = \delta'(a)$. Now the proposition follows. □

The above lemma can be extended to tensor product of finite number of spectral triples with the appropriate assumptions.

Notations. Let us now collect together some of the notations that will be used throughout the paper. The symbol $\mathcal{H}$, with or without subscripts, will denote a Hilbert space. The space of bounded linear operators on $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$ and the space of compact linear

operators on $\mathcal{H}$ will be denoted by $\mathcal{K}(\mathcal{H})$. We will denote by Σ the set $\{1, 2, \ldots, 2\ell + 1\}$ and by Σ_ℓ and $\Sigma_{j,\ell}$ the subsets $\{1, 2, \ldots, \ell + 1\}$ and $\{\ell - j + 1, \ell - j + 2, \ldots, \ell + 1\}$ respectively, where $0 \leq j \leq \ell$.

Let $\Gamma \equiv \Gamma_\Sigma$ denote the set of maps γ from Σ to $\mathbb{Z}$ such that $\gamma_i \in \mathbb{N}$ for all $i \in \Sigma \backslash \{\ell + 1\}$, i.e. $\Gamma_\Sigma = \mathbb{N}^\ell \times \mathbb{Z} \times \mathbb{N}^\ell$. For a subset A of Σ, we will denote by γ_A the restriction $\gamma|_A$ of γ to A. Let Γ_A denote the set $\{\gamma_A : \gamma \in \Gamma\}$ and $\mathcal{H}_A$ be the Hilbert space $\ell_2(\Gamma_A)$. We will denote $\mathcal{H}_\Sigma$ by just $\mathcal{H}$, and $\mathcal{H}_{\Sigma_{j,\ell}}$ by $\mathcal{H}_j$. Thus

$$\mathcal{H}_\Sigma = \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{\ell \text{ copies}} \otimes \ell_2(\mathbb{Z}) \otimes \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{\ell \text{ copies}}, \qquad \mathcal{H}_j = \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{j \text{ copies}} \otimes \ell_2(\mathbb{Z})$$

Note that $\mathcal{H}_j$ and $\mathcal{H}_{\{j\}}$ are different.

Let $A \subseteq \Sigma$. We will denote by $\{e_\gamma\}_\gamma$ the natural orthonormal basis for $\mathcal{H}_A = \ell_2(\Gamma_A)$ and by p_γ the rank one projection $|e_\gamma\rangle\langle e_\gamma|$. For $i \in A$, we will denote by N_i the number operator on the ith coordinate on $\mathcal{H}_A$, i.e.

$$N_i \equiv \sum_\gamma \gamma_i p_\gamma : e_\gamma \mapsto \gamma_i e_\gamma (\text{defined on } \mathcal{H}_A \text{ with } i \in A).$$

We will denote by $|D_A|$ the operator $\sum_{i \in A} |N_i|$ on $\mathcal{H}_A$.

Let F_0 be the following operator on $\ell_2(\mathbb{Z})$:

$$F_0 e_k = \begin{cases} e_k & \text{if } k \geq 0, \\ -e_k & \text{if } k < 0. \end{cases}$$

For $1 \leq j \leq 2\ell + 1$, let V_j be the operator on $\mathcal{H}_{\{j\}}$ defined by

$$V_j := \begin{cases} F_0 & \text{if } j = \ell + 1, \\ I & \text{otherwise.} \end{cases}$$

Let F_A denote the operator $\otimes_{j \in A} V_j$ on $\mathcal{H}_A$ and let $D_A = F_A |D_A|$. Thus

$$D_A e_\gamma = \begin{cases} -\left(\sum_{i \in A} |\gamma_i|\right) e_\gamma & \text{if } \ell + 1 \in A \text{ and } \gamma_{\ell+1} < 0, \\ \left(\sum_{i \in A} |\gamma_i|\right) e_\gamma & \text{otherwise.} \end{cases}$$

We will denote $F_{\Sigma_{j,\ell}}$ by F_j and $D_{\Sigma_{j,\ell}}$ by D_j.

Recall that $\mathcal{H}_{\{j\}}$ is $\ell_2(\mathbb{N})$ if $j \neq \ell + 1$ and is $\ell_2(\mathbb{Z})$ if $j = \ell + 1$. Suppose for each $j \in \Sigma$, $\mathcal{F}_j$ is a subspace of $\mathcal{L}(\mathcal{H}_{\{j\}})$. For $A \subseteq \Sigma$, define

$$\mathcal{F}_{j,A} = \begin{cases} \mathcal{F}_j & \text{if } j \in A, \\ \mathbb{C}.I & \text{if } j \notin A, \end{cases}$$

7

and $\mathcal{F}_A$ to be the tensor product $\otimes_{j \in \Sigma} \mathcal{F}_{j,A}$ in $\mathcal{L}(\mathcal{H}_\Sigma)$ (the type of the tensor product will depend on the specific $\mathcal{F}_j$'s we look at). This tensor product will often be identified with $\otimes_{j \in A} \mathcal{F}_j \subseteq \mathcal{L}(\mathcal{H}_A)$.

On both $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$, we will denote by N the number operator defined by $N e_n = n e_n$ and by S the left shift defined by $S e_n = e_{n-1}$. For $k \in \mathbb{Z}$ (for $k \in \mathbb{N}$ in case of $\ell_2(\mathbb{N})$), let p_k denote the projection $|e_k\rangle\langle e_k|$. We will freely identify $\ell_2(\mathbb{Z})$ with $L_2(\mathbb{T})$. Thus the right shift on $\ell_2(\mathbb{Z})$ will be multiplication by the function $t \mapsto t$ and will be denoted by $\mathbf{z}$. Let $\mathscr{T}$ be the Toeplitz algebra, i.e. the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{N}))$ generated by S. For a positive integer k, we will denote by $\mathscr{T}_k$ the k-fold tensor product of $\mathscr{T}$, embedded in $\mathcal{L}(\ell_2(\mathbb{N}^k))$. Denote by σ the symbol map from $\mathscr{T}$ to $C(\mathbb{T})$ that sends S^* to $\mathbf{z}$ and all compact operators to 0.

3 Torus equivariant spectral triple

3.1 The spectral triple

In this section we recall the spectral triple for the odd dimensional quantum spheres given in [3]. We begin with some known facts about odd dimensional quantum spheres. Let $q \in [0,1]$. The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations (see [10]):

$$
\begin{aligned}
z_i z_j &= q z_j z_i, & 1 \le j < i \le \ell+1, \\
z_i^* z_j &= q z_j z_i^*, & 1 \le i \ne j \le \ell+1, \\
z_i z_i^* - z_i^* z_i + (1-q^2) \sum_{k>i} z_k z_k^* &= 0, & 1 \le i \le \ell+1, \\
\sum_{i=1}^{\ell+1} z_i z_i^* &= 1.
\end{aligned}
$$

We will denote by $\mathcal{A}(S_q^{2\ell+1})$ the *-subalgebra of A_ℓ generated by the z_j's. Note that for $\ell = 0$, the C^*-algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the torus and for $\ell = 1$, it is $C(SU_q(2))$.

There is a natural torus group $\mathbb{T}^{\ell+1}$ action τ on $C(S_q^{2\ell+1})$ as follows. For $w = (w_1, \ldots, w_{\ell+1})$, define an automorphism τ_w by $\tau_w(z_i) = w_i z_i$. Let $Y_{k,q}$ be the following operators on $\mathcal{H}_\ell$:

$$
Y_{k,q} = \begin{cases} \underbrace{q^N \otimes \ldots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1-q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-k \text{ copies}}, & \text{if } 1 \le k \le \ell, \\[3em] \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes S^*, & \text{if } k = \ell+1. \end{cases}
\tag{3.1}
$$

8

Here for $q = 0$, q^N stands for the rank one projection $p_0 = |e_0\rangle\langle e_0|$. Then $\pi_\ell : z_k \mapsto Y_{k,q}$ gives a faithful representation of $C(S_q^{2\ell+1})$ on $\mathcal{H}_\ell$ (see lemma 4.1 and remark 4.5, [10]). We will denote the image $\pi_\ell(C(S_q^{2\ell+1}))$ by $A_\ell(q)$ or by just A_ℓ.

Let $\{e_\gamma : \gamma \in \Gamma_{\Sigma_\ell}\}$ be the standard orthonormal basis for $\mathcal{H}_\ell$. For $w = (w_1, w_2, \cdots, w_{\ell+1})$ we define the unitary U_w on $\mathcal{H}_\ell$ by $U_w(e_\gamma) = w_1^{\gamma_1} w_2^{\gamma_2} \ldots w_{\ell+1}^{\gamma_{\ell+1}} e_\gamma$ where $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_{\ell+1}) \in \Gamma_{\Sigma_\ell}$. Then (π_ℓ, U) is a covariant representation of $(C(S_q^{2\ell+1}), \mathbb{T}^{\ell+1}, \tau)$. Note that $A_\ell \subset \mathscr{T}_\ell \otimes C(\mathbb{T})$.

In [3] all spectral triples equivariant with respect to this covariant representation were characterised and an optimal one was constructed. We recall the following theorem from [3].

Theorem 3.1 ([3]) *Let D_ℓ be the operator $e_\gamma \to d(\gamma)e_\gamma$ on $\mathcal{H}_\ell$ where the d_γ's are given by*

$$d(\gamma) = \begin{cases} \gamma_1 + \gamma_2 + \cdots \gamma_\ell + |\gamma_{\ell+1}| & \text{if } \gamma_{\ell+1} \geq 0 \\ -(\gamma_1 + \gamma_2 + \cdots \gamma_\ell + |\gamma_{\ell+1}|) & \text{if } \gamma_{\ell+1} < 0 \end{cases}$$

Then $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D_\ell)$ is a non-trivial $(\ell+1)$ summable spectral triple. Also D_ℓ commutes with U_w for every $w \in \mathbb{T}^{\ell+1}$.

The operator D_ℓ is optimal i.e. if $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D)$ is a spectral triple such that D commutes with U_w for every w, then there exist positive reals a and b such that $|D| \leq a + b|D_\ell|$.

In this section, we will introduce a dense subalgebra $\mathcal{A}_\ell^\infty$ of $A_\ell(q)$ closed under its holomorphic function calculus and establish regularity of the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$. We will also compute its dimension spectrum.

3.2 The smooth function algebra $\mathcal{A}_\ell^\infty$

In this section we associate a dense Fréchet $*$-subalgebra of $A_\ell(q) = \pi_\ell(C(S_q^{2\ell+1}))$ which is closed under holomorphic functional calculus. We will first show that the C^*-algebra $A_\ell(q)$ is independent of q.

Lemma 3.2 *For any $q \in (0, 1)$, one has $A_\ell(0) = A_\ell(q)$.*

Proof: Let us first show that $A_\ell(q) \subseteq A_\ell(0)$. For $j, k \in \mathbb{N}$ with $j + k \leq \ell$, let

$$X_{j,k} = \underbrace{p_0 \otimes \cdots \otimes p_0}_{j} \otimes \underbrace{q^N \otimes \cdots \otimes q^N}_{k} \otimes S^* \times \underbrace{I \otimes \cdots \otimes I}_{\ell-j-k}.$$

Then $X_{j,0} = Y_{j+1,0}$ for $0 \leq j \leq \ell$, and $X_{j,k} = \sum_{n \in \mathbb{N}} q^n Y_{j,0}^n X_{j+1,k-1} Y_{j,0}^{*n}$. Therefore it follows that $X_{j,k} \in A_\ell(0)$ for $0 \leq j, k \leq \ell$ with $j + k \leq \ell$. Now observe that $Y_{k,q} = (|X_{0,k-1}|^2 - |X_{0,k}|^2)^{\frac{1}{2}}$ for $1 \leq k \leq \ell$ and $Y_{\ell+1,q} = X_{0,\ell}$. Hence $A_\ell(q) \subseteq A_\ell(0)$.

For the other inclusion, we will use the following fact: if B denotes the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{N}))$ generated by the operator $X = (1 - q^{2N})^{\frac{1}{2}} S^*$, then B contains the shift operator S. This is because the operator $|X|$ is invertible and $S^* = X|X|^{-1}$. Using this fact for the first

9

copy of $\ell_2(\mathbb{N})$, since $Y_{1,q} \in A_\ell(q)$, one gets $Y_{1,0} \in A_\ell(q)$. Next assume that $Y_{i,0} \in A_\ell(q)$ for $1 \leq i \leq j-1$, where $2 \leq j \leq \ell$. Then $P_{j-1} := I - \sum_{k=1}^{j-1} Y_{k,0} Y_{k,0}^* \in A_\ell(q)$. Observe that

$$P_{j-1} Y_{j,q} = \underbrace{p_0 \otimes \cdots \otimes p_0}_{j-1} \otimes X \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-j}, \qquad Y_{j,0} = \underbrace{p_0 \otimes \cdots \otimes p_0}_{j-1} \otimes S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-j}.$$

Therefore using the above fact for the jth copy of $\ell_2(\mathbb{N})$, we get $Y_{j,0} \in A_\ell(q)$. Finally, since $Y_{\ell+1,0} = Y_{\ell+1,q}(I - \sum_{k=1}^{\ell} Y_{k,0} Y_{k,0}^*)$, one has $Y_{\ell+1,0} \in A_\ell(q)$. $\qquad\qquad\square$

Let us write α_i for $Y_{i,0}^*$. Note that the C^*-subalgebra of A_ℓ generated by $\alpha_2, \cdots, \alpha_{\ell+1}$ is isomorphic to $A_{\ell-1}$ where the map $a \mapsto p_0 \otimes a$ gives the isomorphism. We define the Fréchet subalgebras $\mathcal{A}_\ell^\infty$ inductively as follows.

The algebra

$$\mathcal{A}_0^\infty := \left\{ \sum_{n \in \mathbb{Z}} a_n \mathbf{z}^n : (a_n) \text{ is rapidly decreasing} \right\}$$

is the algebra of smooth functions on $\mathbb{T}$ together with the increasing family of seminorms $\|\cdot\|_p$ given by $\|(a_n)\|_p = \sum (1 + |n|)^p |a_n|$. Then $\mathcal{A}_0^\infty$ is a dense $*$ Fréchet subalgebra of $A_0 = C(\mathbb{T})$. Note that $\|a\| \leq \|a\|_0$ for $a \in \mathcal{A}_0^\infty$. Now assume that $(\mathcal{A}_{\ell-1}^\infty, \|\cdot\|_m)$ be defined such that

1. the seminorms $\|\cdot\|_m$ are increasing and $(\mathcal{A}_{\ell-1}^\infty, \|\cdot\|_m)$ is a Fréchet algebra,

2. the subalgebra $\mathcal{A}_{\ell-1}^\infty$ is $*$ closed and dense in $A_{\ell-1}$. For every $a \in \mathcal{A}_{\ell-1}^\infty$, one has $\|a^*\|_m = \|a\|_m$,

3. for every $a \in \mathcal{A}_{\ell-1}^\infty$, one has $\|a\| \leq \|a\|_0$ where $\|\cdot\|$ denotes the C^* norm of $A_{\ell-1}$.

Now define

$$\mathcal{A}_\ell^\infty \; := \; \left\{ \sum_{j,k \in \mathbb{N}} \alpha_1^{*j} (p_0 \otimes a_{jk}) \alpha_1^k + \sum_{k \geq 0} \lambda_k \alpha_1^k + \sum_{k > 0} \lambda_{-k} \alpha_1^{*k} \; : \; a_{jk} \in \mathcal{A}_{\ell-1}^\infty, \right.$$
$$\left. \sum_{j,k} (1 + j + k)^n \|a_{jk}\|_m < \infty, \; (\lambda_k) \text{ is rapidly decreasing} \right\}. \qquad (3.2)$$

Let $a := \sum_{j,k} \alpha_1^{*j}(p_0 \otimes a_{jk}) \alpha_1^k + \sum_{k \geq 0} \lambda_k \alpha_1^k + \sum_{k > 0} \lambda_{-k} \alpha_1^{*k}$ be an element of $\mathcal{A}_\ell^\infty$. Define for $m \in \mathbb{N}$, the seminorms $\|a\|_m$ as follows:

$$\|a\|_m = \max_{r,s \leq m} \left(\sum_{j,k} (1 + j + k)^r \|a_{jk}\|_s \right) + \sum_{k \in \mathbb{Z}} (1 + |k|)^m |\lambda_k|.$$

Proposition 3.3 *The pair $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ has the following properties:*

 1. the seminorms $\|\cdot\|_m$ are increasing and $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ is a Fréchet algebra,

2. the subalgebra $\mathcal{A}_\ell^\infty$ is $*$ closed and dense in A_ℓ. For every $a \in \mathcal{A}_\ell^\infty$, one has $\|a^*\|_m = \|a\|_m$,

3. for every $a \in \mathcal{A}_\ell^\infty$, one has $\|a\| \leq \|a\|_0$ where $\|\cdot\|$ denotes the C^* norm of A_ℓ.

Proof: The proof is by induction on ℓ. Parts (2) and (3) and the fact that the seminorms $\|\cdot\|_m$ are increasing follow from the definition and the induction hypothesis. One verifies directly that $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ is a Fréchet algebra using induction and the following relations.

$$\alpha_1 \alpha_1^* = 1$$

$$\alpha_1^{*j}(p_0 \otimes a_{jk})\alpha_1^k \alpha_1^{*r}(p_0 \otimes a_{rs})\alpha_1^s = \delta_{kr}\alpha_1^{*j}(p_0 \otimes a_{jk}a_{rs})\alpha_1^s$$

$$\alpha_1^{*m}\alpha_1^n = \begin{cases} \alpha_1^{*(m-n)} - \sum_{k=0}^{n-1}\alpha_1^{*(m-n+k)}(p_0 \otimes 1)\alpha_1^k & \text{if } m \geq n \\ \alpha_1^{n-m} - \sum_{k=0}^{m-1}\alpha_1^{*k}(p_0 \otimes 1)\alpha_1^{n-m+k} & \text{if } m < n. \end{cases}$$

$\square$

Denote the generators $z_1, z_2, \cdots z_{\ell+1}$ of $C(S_q^{2\ell+1})$ by $z_1^{(\ell+1)}, z_2^{(\ell+1)}, \cdots, z_{\ell+1}^{(\ell+1)}$. Let $\sigma_\ell : C(S_q^{2\ell+1}) \to C(S_q^{2\ell-1})$ be the homomorphism given by $\sigma_\ell(z_{\ell+1}^{(\ell+1)}) = 0$ and $\sigma_\ell(z_i^{(\ell+1)}) = z_i^{(\ell)}$ for $1 \leq i \leq \ell$. Let us denote by the same symbol σ_ℓ the induced homomorphism from A_ℓ to $A_{\ell-1}$. Observe that if one applies the map σ on the ℓth copy of $\mathcal{T}$ in $\mathcal{T}_\ell \otimes C(\mathbb{T})$ followed by evaluation at 1 in the $(\ell+1)$th copy, then the restriction of the resulting map to A_ℓ is precisely σ_ℓ.

Proposition 3.4 *The dense Fréchet $*$-subalgebra $\mathcal{A}_\ell^\infty$ of A_ℓ is closed under holomorphic functional calculus in A_ℓ. Moreover, the algebra $\mathcal{A}_\ell^\infty$ contains the generators $Y_{1,q}^{(\ell+1)}, \cdots, Y_{\ell+1,q}^{(\ell+1)}$.*

Proof: We prove this proposition by induction on ℓ. For $\ell = 0$, by definition $\mathcal{A}_0^\infty = C^\infty(\mathbb{T})$. Hence the proposition is clear in this case. Now assume that the algebra $\mathcal{A}_{\ell-1}^\infty$ is closed under holomorphic functional calculus in $A_{\ell-1}$ and contains $Y_{1,q}^{(\ell)}, \cdots, Y_{\ell,q}^{(\ell)}$. The homomorphism $\sigma_\ell : A_\ell \to A_{\ell-1}$ gives the following exact sequence

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T}) \longrightarrow A_\ell \longrightarrow A_{\ell-1} \longrightarrow 0$$

One also has at the smooth algebra level the "sub" extension

$$0 \longrightarrow \mathcal{S}(\ell_2(\mathbb{N}^\ell)) \otimes C^\infty(\mathbb{T}) \longrightarrow \mathcal{A}_\ell^\infty \longrightarrow \mathcal{A}_{\ell-1}^\infty \longrightarrow 0.$$

Since $\mathcal{S}(\ell_2(\mathbb{N}^\ell)) \otimes C^\infty(\mathbb{T}) \subset \mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T})$ and $\mathcal{A}_{\ell-1}^\infty \subset A_{\ell-1}$ are closed under the respective holomorphic functional calculus, it follows from theorem 3.2, part 2, [13] that $\mathcal{A}_\ell^\infty$ is spectrally invariant in A_ℓ. Since $\|a\| \leq \|a\|_0$ for all $a \in \mathcal{A}_\ell^\infty$, it follows that the Fréchet topology of $\mathcal{A}_\ell^\infty$ is finer than the norm topology. Therefore $\mathcal{A}_\ell^\infty$ is closed under holomorphic functional calculus in A_ℓ. Observe that for $i \geq 2$, we have $Y_{i,q}^{(\ell+1)} = \sum_{n\geq 0} q^n \alpha_1^{*n}(p_0 \otimes Y_{i-1,q}^{(\ell)})\alpha_1^n$. Hence $Y_{i,q}^{(\ell+1)} \in \mathcal{A}_\ell^\infty$ for $i = 2, \cdots, \ell+1$. Also note that $q^N \otimes I = \sum_{n\geq 0} q^n \alpha_1^{*n}(p_0 \otimes 1)\alpha_1$. Since $\mathcal{A}_\ell^\infty$

is closed under holomorphic functional calculus, it follows that $\sqrt{1 - q^{2N+2}} \otimes I \in \mathcal{A}_\ell^\infty$. As $Y_{1,q}^{(\ell+1)} = \alpha_1^*(\sqrt{1 - q^{2N+2}} \otimes I)$ it follows that $Y_{1,q}^{(\ell+1)} \in \mathcal{A}_\ell^\infty$. This completes the proof. $\qquad\square$

Next we proceed to prove that the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and compute its dimension spectrum. The proof is by induction. We start with the case $\ell = 0$ to start the induction.

3.3   The case $\ell = 0$

For $\ell = 0$, the spectral triple $(\mathcal{A}_0^\infty, \mathcal{H}_0, D_0)$ is unitarily equivalent to the spectral triple $(C^\infty(\mathbb{T}), L_2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta})$. For $f \in C^\infty(\mathbb{T})$ one has $[D_0, f] = \frac{1}{i}f'$. Let (e_k) be the standard orthonormal basis and let p_k be the projection onto e_k. Let $F_0 := sign(D_0)$. Note that $[F_0, \mathbf{z}] = 2p_0\mathbf{z}$ and hence by induction $[F_0, \mathbf{z}^n] = 2\sum_{k=0}^{n-1} p_k\mathbf{z}^n p_{k-n}$ for $n \geq 0$. Thus $[F_0, \mathbf{z}^n]$ is smoothing for $n \geq 0$. Also $\||D_0|^r[F_0, \mathbf{z}^n]|D_0|^s\| \leq 2(1 + n)^{r+s+1}$. Since $[F_0, \mathbf{z}^{-|n|}]^* = -[F_0, \mathbf{z}^{|n|}]$, it follows that $[F_0, \mathbf{z}^n] \in OP^{-\infty}$ for every n. Also $\|[F_0, \mathbf{z}^n]\|_{r,s} \leq 2(1 + |n|)^{r+s+1}$. Hence we observe that $[F_0, f] \in OP^{-\infty}$ and $\|[F_0, f]\|_{r,s} \leq 2\|f\|_{r+s+1}$. Let δ be the unbounded derivation $[|D_0|, \cdot]$.

Lemma 3.5 Let $\mathcal{B} := \{f_0 + f_1F_0 + R : f_0, f_1 \in C^\infty(\mathbb{T}), R \in OP_{D_0}^{-\infty}\}$. Then

1. If $f_0 + f_1F_0$ is smoothing then $f_0 = f_1 = 0$. Hence $\mathcal{B}$ is isomorphic to the direct sum $C^\infty(\mathbb{T}) \oplus C^\infty(\mathbb{T}) \oplus OP_{D_0}^{-\infty}$. We give $\mathcal{B}$ the Fréchet space structure coming from this decomposition. This topology on $\mathcal{B}$ is generated by the seminorms $(\|\cdot\|_m)_{m\in\mathbb{N}}$ which are defined by $\|f_0 + f_1F_0 + R\|_m := \|f_0\|_m + \|f_1\|_m + \sum_{r+s\leq m}\|R\|_{r,s}$.

2. The vector space $\mathcal{B}$ is closed under δ and the derivation $[D_0, \cdot]$.

3. For every $b \in \mathcal{B}$, $[F_0, b] \in OP^{-\infty}$. Also the map $b \to [F_0, b] \in OP^{-\infty}$ is continuous. The derivations δ and $[D_0, \cdot]$ are continuous.

4. The vector space $\mathcal{B}$ is an algebra and contains $C^\infty(\mathbb{T})$.

Proof: First observe that a bounded operator T on $\ell_2(\mathbb{Z})$ is smoothing if and only if $(\langle Te_m, e_n\rangle)_{m,n}$ is rapidly decreasing. Now suppose that $R := f_0 + f_1F_0$ be smoothing. Fix an integer r. Observe that $\langle R(e_n), e_{r+n}\rangle$ converges to $\hat{f}_0(r) + \hat{f}_1(r)$ as $n \to +\infty$ and converges to $\hat{f}_0(r) - \hat{f}_1(r)$ as $n \to -\infty$. But since R is smoothing it follows that $\hat{f}_0(r) + \hat{f}_1(r) = 0 = \hat{f}_0(r) - \hat{f}_1(r)$. Hence $\hat{f}_0(r) = \hat{f}_1(r) = 0$ for every integer r. Thus $f_0 = f_1 = 0$. This proves part (1).

Parts (2), (3) and (4) follow from the observations that $[D_0, f] = \frac{1}{i}f'$, $[F_0, f] \in OP^{-\infty}$, $\|[F_0, f]\|_{r,s} \leq 2\|f\|_{r+s+1}$ and $\delta(b) = [D_0, b]F_0 + D_0[F_0, b]$. This completes the proof. $\qquad\square$

In particular, it follows from parts (2) and (4) of the above lemma that the spectral triple $(\mathcal{A}_0^\infty, \mathcal{H}_0, D_0)$ is regular.

Let $\mathcal{E}$ be the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{Z}))$ generated by $C(\mathbb{T})$ and F_0. Note that the algebra $\mathcal{B}$ plays the role of smooth function subalgebra for the C^*-algebra $\mathcal{E}$. Therefore $\mathcal{E}^\infty$ will stand for the algebra $\mathcal{B}$.

12

3.4 Regularity and the dimension spectrum

In this subsection we prove regularity and calculate the dimension spectrum for the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$. The proof is by induction on ℓ. Let us denote the derivation $[|D_\ell|, \cdot]$ by δ_ℓ and let F_ℓ stand for the sign of the operator D_ℓ. Observe that $F_\ell = 1^{\otimes \ell} \otimes F_0 = 1 \otimes F_{\ell-1}$.

Proposition 3.6 *Let $\mathcal{B}_\ell := \{A_0 + A_1 F_\ell + R : A_0, A_1 \in \mathcal{A}_\ell^\infty, R \in OP^{-\infty}\}$. Then*

1. *if $A_0 + A_1 F_\ell$ is smoothing then $A_0 = A_1 = 0$. Hence $\mathcal{B}_\ell$ is isomorphic to the direct sum $\mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus OP^{-\infty}$. Equip $\mathcal{B}_\ell$ with the Fréchet space structure coming from this decomposition. This topology on $\mathcal{B}_\ell$ is induced by the seminorms $(\|\cdot\|_m)_{m\in\mathbb{N}}$ which are defined by $\|A_0 + A_1 F_\ell + R\|_m := \|A_0\|_m + \|A_1\|_m + \sum_{r+s\le m} \|R\|_{r,s}$.*

2. *For every $b \in \mathcal{B}_\ell$, $[F_\ell, b] \in OP^{-\infty}$. Also the map $b \to [F_\ell, b] \in OP^{-\infty}$ is continuous.*

3. *The vector space $\mathcal{B}_\ell$ is closed under the derivations δ_ℓ and $[D_\ell, \cdot]$. Moreover the derivations δ_ℓ and $[D_\ell, \cdot]$ are continuous.*

4. *The vector space $\mathcal{B}_\ell$ is an algebra and contains $\mathcal{A}_\ell^\infty$.*

Proof: The proof is by induction on ℓ. For $\ell = 0$, the proposition is just lemma 3.5. Now assume that the proposition is true for $\ell - 1$. Suppose that $A_0 + A_1 F_\ell$ is smoothing for some $A_0, A_1 \in \mathcal{A}_\ell^\infty$. Then $A_0 + A_1 F_\ell \in \mathscr{T}_\ell \otimes \mathcal{E}$ and $A_0 + A_1 F_\ell$ is compact. Therefore $(\sigma \otimes id)(A_0 + A_1 F_\ell) = 0$. Now let

$$A_i = \sum_{j,k\ge 0} \alpha_1^{*j}(p_0 \otimes a_{jk}^{(i)})\alpha_1^k + \sum_{k\ge 0} \lambda_k^{(i)}\alpha_1^k + \sum_{k>0} \lambda_{-k}^{(i)}\alpha_1^{*k}$$

for $i = 0, 1$. Let $f_i(z) = \sum_{k\in\mathbb{Z}} \lambda_k^{(i)} z^k$ for $i = 0, 1$. Now $(\sigma \otimes id)(A_0 + A_1 F_\ell) = f_0 \otimes I + f_1 \otimes F_{\ell-1}$. So we have $f_0 \otimes I + f_1 \otimes F_{\ell-1} = 0$. Writing $F_\ell = 2P_\ell - I$, it follows that $(f_0 + f_1) \otimes P_{\ell-1} + (f_0 - f_1) \otimes (1 - P_{\ell-1}) = 0$. Hence $f_0 = f_1 = 0$. This shows that $\lambda_k^{(i)} = 0$ for $i = 0, 1$. Let $b_{jk} = a_{jk}^0 + a_{jk}^1 F_{\ell-1}$. Since $R := A_0 + A_1 F_\ell$ is smoothing, it follows that for every j, k, the matrix entries $\langle e_{(j,\gamma)}, R(e_{(k,\gamma')})\rangle$ are rapidly decreasing in (γ, γ'). Hence b_{jk} is smoothing for every j, k. By induction hypothesis $a_{jk}^{(i)} = 0$ for every $j, k \ge 0$ and for $i = 0, 1$. Thus $A_0 = A_1 = 0$. This proves part (1).

Observe that

$$\delta_\ell(\alpha_1) = -\alpha_1, \qquad |D_\ell|^r \alpha_1^{*k} = \alpha_1^{*k}(|D_\ell| + k)^r, \qquad \alpha_1^k |D_\ell|^s = (|D_\ell| + k)^s \alpha_1^k.$$

Also F_ℓ commutes with α_1. To prove (2), it is enough to show that $[F_\ell, a]$ is smoothing for every $a \in \mathcal{A}_\ell^\infty$ and the map $a \mapsto [F_\ell, a]$ is continuous. Let

$$a = \sum_{m,n\ge 0} \alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \sum_{m\ge 0} \lambda_m \alpha_1^m + \sum_{m>0} \lambda_{-m}\alpha_1^{*m}$$

13

be an element in $\mathcal{A}_\ell^\infty$. Then $[F_\ell, a] = \sum_{m,n \geq 0} \alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n$. By induction hypothesis, it follows that $p_0 \otimes [F_{\ell-1}, a_{mn}]$ is smoothing for every $m, n \geq 0$. Since $(OP_{D_\ell}^{-\infty}, \|\cdot\|_{r,s})$ is a Fréchet space, to show that $[F_\ell, a]$ is smoothing it is enough to show that the infinite sum $\sum_{m,n\geq 0} \alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n$ converges absolutely in every seminorm $\|\cdot\|_{r,s}$. Now observe that

$$\sum_{m,n} |D_\ell|^r \alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n |D_\ell|^s = \sum_{m,n} \alpha_1^{*m}(|D_\ell| + m)^r (p_0 \otimes [F_{\ell-1}, a_{mn}])(|D_\ell| + n)^s \alpha_1^n.$$

(3.3)

Since the map $a' \in \mathcal{A}_{\ell-1}^\infty \mapsto [F_{\ell-1}, a'] \in OP^{-\infty}$ is continuous, there exist $p \in \mathbb{N}$ and $C_p > 0$ such that $\|[F_{\ell-1}, a']\|_{i,j} \leq C_p \|a'\|_p$ for every $a' \in \mathcal{A}_{\ell-1}^\infty$ and for $i, j \leq max\{r, s\}$. Hence by equation (3.3), it follows that

$$\sum_{m,n} \|\alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n\|_{r,s} \leq \sum_{m,n} \sum_{i=0}^r \sum_{j=0}^s \binom{r}{i}\binom{s}{j} m^{r-i} n^{s-j} \|[F_{\ell-1}, a_{mn}]\|_{i,j}$$

$$\leq \sum_{i=0}^r \sum_{j=0}^s \binom{r}{i}\binom{s}{j} C_p \left(\sum_{m,n} m^r n^s \|a_{mn}\|_p \right).$$

This shows that $[F_\ell, a]$ is smoothing and the above inequality also shows that for every $r, s \geq 0$, there exists $t \geq 0$ and a $C_t > 0$ such that $\|[F_\ell, a]\|_{r,s} \leq C_t \|a\|_t$. Hence the map $a \mapsto [F_\ell, a]$ is continuous. This proves (2).

To show (3), it is enough to show that the map $a \mapsto \delta_\ell(a)$ from $\mathcal{A}_\ell^\infty$ to $\mathcal{B}_\ell$ makes sense and is continuous. We will use the fact that the unbounded derivation δ_ℓ is a closed derivation. Let $a = \sum_{m,n \geq 0} \alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \sum_{m \geq 0} \lambda_m \alpha_1^m + \sum_{m > 0} \lambda_{-m} \alpha_1^{*m}$ be an element in $\mathcal{A}_\ell^\infty$. Since α_1 and $p_0 \otimes a_{mn} \in Dom(\delta_\ell)$ it follows that each of the terms in the infinite sum is an element in $Dom(\delta_\ell)$. Hence in order to show $a \in Dom(\delta_\ell)$, it is enough to show that the sum

$$\sum_{m,n} \delta_\ell(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n) + \sum_{m \geq 0} \lambda_m \delta_\ell(\alpha_1^m) + \sum_{n > 0} \lambda_{-n} \delta_\ell(\alpha_1^{*n})$$

converges. Observe that $\delta_\ell(\alpha_1^{*m}) = m\alpha_1^{*m}$, $\delta_\ell(\alpha_1^n) = -n\alpha_1^n$ and

$$\delta_\ell\big(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n\big) = (m-n)\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \alpha_1^{*m}(p_0 \otimes \delta_{\ell-1}(a_{mn}))\alpha_1^n$$

Since $\delta_{\ell-1}$ is continuous, it follows that $\|\delta_{\ell-1}(a_{mn})\|$ is rapidly decreasing where $\|\cdot\|$ is the operator norm. (Note that for $b \in \mathcal{B}_\ell$, one has $\|b\| \leq \|b\|_0$.) Hence the infinite sum

$$\sum_{m,n} \delta_\ell\big(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n\big) + \sum_{m \geq 0} \lambda_m \delta_\ell(\alpha_1^m) + \sum_{n > 0} \lambda_{-n} \delta_\ell(\alpha_1^{*n})$$

converges absolutely in the operator norm. Therefore $a \in Dom(\delta_\ell)$. Since $\delta_{\ell-1}$ is continuous for every r there exists p and C_p such that $\|\delta_{\ell-1}(a')\|_r \leq C_p \|a'\|_p$. Write $\delta_{\ell-1}(a_{mn})$ as $\delta_{\ell-1}(a_{mn}) =$

14

$a'_{mn} + a''_{mn}F_\ell + R_{mn}$. Let

$$A_0 = \sum_{m,n} \alpha_1^{*m}(p_0 \otimes ((m-n)a_{mn} + a'_{mn}))\alpha_1^n + \sum_{m \geq 0} m\lambda_m\alpha_1^m + \sum_{n > 0}(-n)\lambda_{-n}\alpha_1^{*n},$$

$$A_1 = \sum_{m,n} \alpha_1^{*m}(p_0 \otimes a''_{mn})\alpha_1^n,$$

$$R = \sum_{m,n} \alpha_1^{*m}(p_0 \otimes R_{mn})\alpha_1^n.$$

Then $\delta_\ell(a) = A_0 + A_1 F_\ell + R$. In every seminorm of $\mathcal{A}_{\ell-1}^\infty$ the double sequence (a'_{mn}) and (a''_{mn}) are rapidly decreasing. Also R_{mn} is rapidly decreasing in every seminorm of $OP_{D_\ell}^{-\infty}$. Hence $A_0, A_1 \in \mathcal{A}_\ell^\infty$ and as in the proof of (2), it follows that R is smoothing and given r, s there exists t and C_t such that $\|R\|_{r,s} \leq C_t\|a\|_t$. Fix an $r \geq 0$ and choose $t > 1 + r$ and $C_t > 1$ such that $\|\delta_{\ell-1}(a')\|_r \leq C_t\|a'\|_t$ for every $a' \in \mathcal{A}_{\ell-1}^\infty$. Now $\|A_0\|_r \leq C_t\|a\|_t$ and $\|A_1\|_r \leq C_t\|a\|_t$. This shows that the map $a \to \delta_\ell(a) \in \mathcal{B}_\ell$ is continuous. Since $[D_\ell, b] = \delta_\ell(b)F_\ell + |D_\ell|[F_\ell, b]$, the second part of (3) follows as $[F_\ell, b]$ is smoothing by (2). This proves (3).

Part (4) follows from (2) and (3). $\qquad\square$

We next prove a lemma that will be crucial in the computation of the dimension spectrum. For an r tuple $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$, we will write $|n|$ for $\sum_{i=1}^r n_i$. For $r = 0$, we let $\mathbb{N}^0 = 0$.

Lemma 3.7 *Let $r \geq 0$ and $s \geq 1$ be integers. Let $(a(n))_{n \in \mathbb{N}^r}$ be rapidly decreasing. Then the function*

$$\xi(z) := \sum_{\substack{n \in \mathbb{N}^r, m \in \mathbb{N}^s \\ |n|+|m| \geq 1}} \frac{a(n)}{(|n| + |m|)^z}$$

is meromorphic with simple poles at $\{1, 2, \cdots, s\}$ and $\mathrm{Res}_{z=s}\xi(z) = \frac{1}{(s-1)!}\sum_n a(n)$.

Proof: First observe that for $\mathrm{Re}\, z > r + s$,

$$\xi(z) = \sum_{N \geq 1} \frac{1}{N^z}\left(\sum_{|n|+|m|=N} a(n)\right)$$

$$= \sum_{N \geq 1} \frac{1}{N^z}\left(\sum_{|n| \leq N} a(n) \sum_{m:|m|=N-|n|} 1\right)$$

$$= \sum_{N \geq 1} \frac{1}{N^z}\left(\sum_{|n| \leq N} a(n)\binom{N - |n| + s - 1}{s - 1}\right).$$

Note that for a function $(b(n))_{n \in \mathbb{N}^r}$ of rapid decay, the sequence $\left(\sum_{|n| \geq N} b(n)\right)_{N \in \mathbb{N}}$ is of rapid decay. Now $\binom{N-|n|+s-1}{s-1} = \sum_{k=0}^{s-1} g_k(n)N^k$ where $g_k(n)$ is a polynomial in $(n_1, n_2, \ldots, n_r)$ and

15

$g_{s-1}(n) = \frac{1}{(s-1)!}$. Hence modulo a holomorphic function $\xi(z) = \sum_{k=0}^{s-1}(\sum_n g_k(n)a(n))\zeta(z-k)$. Now the result follows from the fact that $\zeta(z)$ is meromorphic with a simple pole at $z = 1$ with residue 1. $\qquad\square$

We will next prove that the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and has discrete dimension spectrum with simple poles at $\{1, 2, \cdots, \ell+1\}$.

Remark 3.8 Recall that the unitaries U_w for $w = (w_1, w_2, \cdots, w_{\ell+1}) \in \mathbb{T}^{\ell+1}$ are given by $U_w e_\gamma = w_1^{\gamma_1} w_2^{\gamma_2} \cdots w_{\ell+1}^{\gamma_{\ell+1}} e_\gamma$. A bounded operator T on $\mathcal{H}_\ell$ is said to be homogeneous of degree $(m_1, m_2, \cdots, m_{\ell+1})$ if $U_w T U_w^* = w_1^{m_1} w_2^{m_2} \cdots w_{\ell+1}^{m_{\ell+1}} T$. If T is homogeneous of degree $(m_1, m_2, \cdots, m_{\ell+1}) \neq (0, \ldots, 0)$ then $Trace(T|D|^{-z}) = 0$ if $Re(z) > \ell+1$ since U_w's commute with the operator $|D_\ell|$.

Proposition 3.9 *The spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and has $\{1, 2, \cdots, \ell+1\}$ as the dimension spectrum with only simple poles.*

Proof: Regularity of the spectral triple follows from proposition 3.6. We now prove that for $b \in \mathcal{B}_\ell$, the function $Trace(b|D|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, \ell+1\}$. Since $Trace\,(b|D|^{-z})$ is holomorphic for $b \in OP^{-\infty}$, we need only to show that for $a \in \mathcal{A}_\ell^\infty$, the functions $Trace(a|D|^{-z})$ and $Trace(aF_\ell|D|^{-z})$ extend to meromorphic functions with simple poles at $\{1, 2, \cdots, \ell+1\}$. Now any element $a \in \mathcal{A}_\ell^\infty$ can be written as $a = a^0 + a^1$ where a^0 is homogeneous of degree 0 and a^1 is an infinite sum of homogeneous elements of non zero degrees. Hence by remark 3.8, $Trace(a|D|^{-z}) = Trace(a^0|D|^{-z})$ and $Trace(aF_\ell|D|^{-z}) = Trace(a^0 F_\ell|D|^{-z})$. Thus it is enough to consider the functions $Trace(a|D|^{-z})$ and $Trace(aF_\ell|D|^{-z})$ where a is homogeneous of degree 0.

It is easy to see that the set of homogeneous elements of degree 0 in $\mathcal{A}_\ell^\infty$ is

$$\left\{ \sum_{i=0}^{\ell} \left(\sum_{n \in \mathbb{N}^i} \lambda_n^i (p_{n_1} \otimes p_{n_2} \otimes \cdots p_{n_i} \otimes 1) \right) : (\lambda_n^i) \text{ is of rapid decay for all } i \right\}$$

where $p_k = S^{*k} p_0 S^k$. Let $a = \sum_{i=0}^{\ell}(\sum_n \lambda_n^i (p_{n_1} \otimes p_{n_2} \otimes \cdots p_{n_i} \otimes 1))$ be a homogeneous element of degree 0 in $C^\infty(S_q^{2\ell+1})$. Then

$$Trace(a|D|^{-z}) = 2\sum_{i=0}^{\ell} \sum_{\substack{n \in \mathbb{N}^i, t \in \mathbb{N} \\ m \in \mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n| + |m| + t)^z} + \sum_{i=0}^{\ell} \sum_{\substack{n \in \mathbb{N}^i \\ m \in \mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n| + |m|)^z}.$$

Now $\sum_{n \in \mathbb{N}^\ell} \frac{\lambda_n^\ell}{(|n|)^z}$ is holomorphic and hence modulo a holomorphic function

$$Trace(a|D|^{-z}) = 2\sum_{i=0}^{\ell} \left(\sum_{\substack{n \in \mathbb{N}^i, t \in \mathbb{N} \\ m \in \mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n| + |m| + t)^z} \right) + \sum_{i=0}^{\ell-1} \left(\sum_{\substack{n \in \mathbb{N}^i \\ m \in \mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n| + |m|)^z} \right)$$

16

It follows from lemma 3.7 that $Trace(a|D|^{-z}$ is meromorphic with simple poles at $\{1, 2, \cdots, \ell + 1\}$. Similarly one can show that $Trace(aF_\ell|D|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, \ell\}$. Fix $0 \leq i \leq \ell+1$. Let $(\lambda_n)_{n \in \mathbb{N}^i}$ be such that $\sum_n \lambda_n = 1$. Let $a = \sum_{n \in \mathbb{N}^i} \lambda_n(p_{n_1} \otimes p_{n_2} \otimes p_{n_i} \otimes 1)$. Then one has $Res_{z=\ell+1-i}Trace(a|D|^{-z}) = \frac{2}{(\ell-i)!}$ by lemma 3.7 and by the above equation. Hence every $k \in \{1, 2, \cdots, \ell + 1\}$ is in the dimension spectrum. This completes the proof. $\square$

4 $SU_q(\ell + 1)$ Equivariant spectral triple

4.1 Left multiplication operators

Let us recall from [4] some basic facts on representations of $C(SU_q(\ell + 1))$ on $L_2(SU_q(\ell + 1))$ by left multiplication. Irreducible unitary representations of the quantum group $SU_q(\ell+1)$ are indexed by Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$ where $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell+1} = 0$ (Theorem 1.5, [15]). Denote by u^λ the irreducible unitary indexed by λ. Basis elements of the Hilbert space $\mathcal{H}_\lambda$ on which u^λ acts can be parametrized by arrays of the form

$$\mathbf{r} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,\ell} & r_{1,\ell+1} \\ r_{21} & r_{22} & \cdots & r_{2,\ell} \\ & & \cdots & \\ r_{\ell,1} & r_{\ell,2} & & \\ r_{\ell+1,1} & & & \end{pmatrix},$$

where r_{ij}'s are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \ldots, \ell + 1$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j, and the top row coincides with λ. These are known as Gelfand-Tsetlin tableaux, to be abreviated as GT tableaux now onwards. Let $\{e(\lambda, \mathbf{r}) : \mathbf{r}$ is a GT tableaux with top row $\lambda\}$ be an orthonormal basis for $\mathcal{H}_\lambda$. Denote the matrix entries of u^λ with respect to this basis by $u^\lambda_{\mathbf{r},\mathbf{s}}$. Note that the generators u_{ij} of the C^*-algebra $C(SU_q(\ell + 1))$ are the matrix entries of the irreducible $\mathbb{1} = (1, 0, \ldots, 0)$. The collection $\{u^\lambda_{\mathbf{r},\mathbf{s}} : \lambda, \mathbf{r}, \mathbf{s}\}$ form a complete orthogonal set of vectors in $L_2(SU_q(\ell + 1))$. Denote by $e^\lambda_{\mathbf{r},\mathbf{s}}$, or by $e_{\mathbf{r},\mathbf{s}}$ for short (as $\mathbf{r}$ and $\mathbf{s}$ specify λ), the normalized $u^\lambda_{\mathbf{r},\mathbf{s}}$'s, i.e. $e_{\mathbf{r},\mathbf{s}} = \|u^\lambda_{\mathbf{r},\mathbf{s}}\|^{-1}u^\lambda_{\mathbf{r},\mathbf{s}}$. Then $\{e_{\mathbf{r},\mathbf{s}} : \mathbf{r}, \mathbf{s}\}$ form a complete orthonormal basis for $L_2(SU_q(\ell + 1))$.

Let ρ be the half-sum of positive roots of $sl(\ell + 1)$ and $\lambda(\mathbf{r})$ is the weight of the weight vector $e(\lambda, \mathbf{r})$. Let F_λ be the unique intertwiner in $\mathrm{Mor}\,(u^\lambda, (u^\lambda)^{cc})$ with trace $F_\lambda = \mathrm{trace}\,F_\lambda^{-1}$ (here for a representation u, its contragradient representation is denoted by u^c; see [11] for details). Then one has $\|u^\lambda_{\mathbf{rs}}\| = d_\lambda^{-\frac{1}{2}}q^{-\psi(\mathbf{r})}$, where

$$\psi(\mathbf{r}) = (\rho, \lambda(\mathbf{r})) = -\frac{\ell}{2}\sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1}\sum_{j=1}^{\ell+2-i} r_{ij}, \qquad d_\lambda = \mathrm{trace}\,F_\lambda = \sum_{\mathbf{r}:r_1=\lambda} q^{2\psi(\mathbf{r})}. \qquad (4.4)$$

Write

$$\kappa(\mathbf{r}, \mathbf{m}) = d_\lambda^{\frac{1}{2}}d_\mu^{-\frac{1}{2}}q^{\psi(\mathbf{r})-\psi(\mathbf{m})}. \qquad (4.5)$$

17

From equation (4.19) in [4], we have

$$\pi(u_{ij})e^{\lambda}_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})\kappa(\mathbf{r},\mathbf{m})e^{\mu}_{\mathbf{mn}}, \tag{4.6}$$

where C_q denote the Clebsch Gordon coefficients.

For our subsequent analysis, we will compute the quantities $C_q(i,\mathbf{r},\mathbf{s})$ and $\kappa(\mathbf{r},\mathbf{m})$ appearing in the above formula. We will use the formulae given in ([11], pp. 220), keeping in mind that for our case (i.e. for $SU_q(\ell+1)$), the top right entry of the GT tableaux is zero.

For a positive integer j with $1 \le j \le \ell+1$, let

$$\mathbb{M}_j := \{(m_1,m_2,\ldots,m_j) \in \mathbb{N}^j : 1 \le m_i \le \ell+2-i \text{ for } 1 \le i \le j\} \tag{4.7}$$

For $M = (m_1,m_2,\ldots,m_i) \in \mathbb{M}_i$, denote by $M(\mathbf{r})$ the tableaux $\mathbf{s}$ defined by

$$s_{jk} = \begin{cases} r_{jk}+1 & \text{if } k = m_j, 1 \le j \le i, \\ r_{jk} & \text{otherwise.} \end{cases} \tag{4.8}$$

With this notation, observe now that $C_q(i,\mathbf{r},\mathbf{s})$ will be zero unless $\mathbf{s}$ is $M(\mathbf{r})$ for some $M \in \mathbb{M}_i$. (One has to keep in mind though that not all tableaux of the form $M(\mathbf{r})$ is a valid GT tableaux)

From ([11], pp. 220), we have

$$C_q(i,\mathbf{r},M(\mathbf{r})) = \prod_{a=1}^{i-1} R(\mathbf{r},a,m_a,m_{a+1}) \times R'(\mathbf{r},i,m_i) \tag{4.9}$$

where e_k stands for a vector (in the appropriate space) whose kth coordinate is 1 and the rest are all zero, and

$$R(\mathbf{r},a,j,k) = \text{sign}\,(k-j)\,q^{\frac{1}{2}\left(-r_{aj}+r_{a+1,k}-k+j\right)}$$

$$\times \left(\prod_{\substack{i=1 \\ i \ne j}}^{\ell+2-a} \frac{[r_{a,i}-r_{a+1,k}-i+k]_q}{[r_{a,i}-r_{a,j}-i+j]_q} \prod_{\substack{i=1 \\ i \ne k}}^{\ell+1-a} \frac{[r_{a+1,i}-r_{a,j}-i+j-1]_q}{[r_{a+1,i}-r_{a+1,k}-i+k-1]_q}\right)^{\frac{1}{2}} \tag{4.10}$$

$$R'(\mathbf{r},a,j) = q^{\frac{1}{2}\left(1-j+\sum_{i=1}^{\ell+1-a} r_{a+1,i}-\sum_{\substack{i=1 \\ i \ne j}}^{\ell+2-a} r_{a,i}\right)}$$

$$\times \left(\frac{\prod_{i=1}^{\ell+1-a}[r_{a+1,i}-r_{aj}-i+j-1]_q}{\prod_{\substack{i=1 \\ i \ne j}}^{\ell+2-a}[r_{a,i}-r_{aj}-i+j]_q}\right)^{\frac{1}{2}}, \tag{4.11}$$

where for an integer n, $[n]_q$ denotes the q-number $(q^n-q^{-n})/(q-q^{-1})$ and $\text{sign}\,(k-j)$ is 1 if $k \ge j$ and is -1 if $k < j$.

Remark 4.1 Let us look at the denominators in the above expressions. The integers $r_{a,i}-r_{a,j}$ and $j-i$ are of the same sign. Therefore for $i \ne j$, the quantity $r_{a,i}-r_{a,j}-i+j$ is nonzero.

18

Similarly $r_{a+1,i} - r_{a+1,k}$ and $k-i$ are of the same sign. So if $i \neq k$, then $r_{a+1,i} - r_{a+1,k} - i + k - 1$ can be zero only when $r_{a+1,i} = r_{a+1,k}$ and $k = i+1$. Now if $\mathbf{r}$ and $M(\mathbf{r})$ are GT tableaux, then $M(\mathbf{r})_{a+1,m_{a+1}} = r_{a+1,m_{a+1}} + 1$ and $M(\mathbf{r})_{a+1,i} = r_{a+1,i}$ for $i \neq m_{a+1}$. Therefore if $m_{a+1} = i+1$, then $r_{a+1,i} - (r_{a+1,m_{a+1}} + 1) \geq 0$, i.e. $r_{a+1,i} - r_{a+1,m_{a+1}} \geq 1$. Hence $r_{a+1,i} - r_{a+1,m_{a+1}} - i + m_{a+1} - 1 \geq 1$. In other words, all the q-numbers appearing in the denominator in equation (4.9) are nonzero. Thus no problem arises from division by zero.

Remark 4.2 This is essentially a repetition of remark 4.1 of [4]. The formulae (4.10) and (4.11) are obtained from equations (45) and (46), page 220, [11] by replacing q with q^{-1}. Equation (45) is a special case of the more general formula (48), page 221, [11]. However, there is a small error in equation (48) there. The correct form can be found in equations (3.1, 3.2a, 3.2b) in [1]. Here we have incorporated that correction in equations (4.10) and (4.11).

We next compute the quantities $R(\mathbf{r}, a, j, k)$ and $R'(\mathbf{r}, a, j)$.

For a positive integer n, denote by $Q(n)$ the number $(1 - q^{2n})^{1/2}$. Then for any two integers m and n, one has

$$\left| \frac{[m]_q}{[n]_q} \right| = q^{-|m|+|n|} \left(\frac{Q(|m|)}{Q(|n|)} \right)^2.$$

The next two lemmas are obtained from equations (4.10) and (4.11) using the above equality repeatedly and the fact that $r_{a,i} \geq r_{a+1,i} \geq r_{a,i+1}$ for all a and i.

Lemma 4.3 *For a GT tableaux* $\mathbf{r} = (r_{ab})$, *denote by* $H_{ab}(\mathbf{r})$ *and* $V_{ab}(\mathbf{r})$ *the following differences:* $H_{ab}(\mathbf{r}) := r_{a+1,b} - r_{a,b+1}$ *and* $V_{ab}(\mathbf{r}) := r_{ab} - r_{a+1,b}$. *Then one has*

$$R(\mathbf{r}, a, j, k) = sign(k - j) q^{P(\mathbf{r}, a, j, k) + S(\mathbf{r}, a, j, k)} L(\mathbf{r}, a, j, k), \tag{4.12}$$

where

$$P(\mathbf{r}, a, j, k) = \sum_{j \wedge k \leq i < j \vee k} H_{ai}(\mathbf{r}) + 2 \sum_{k < i < j} V_{ai}(\mathbf{r}), \tag{4.13}$$

$$S(\mathbf{r}, a, j, k) = \begin{cases} 2(j - k - 1) + 1 & \text{if } j > k, \\ 0 & \text{if } j \leq k, \end{cases} \tag{4.14}$$

$$L(\mathbf{r}, a, j, k) = \prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} \frac{Q(|r_{a,i} - r_{a+1,k} - i + k|)}{Q(|r_{a,i} - r_{a,j} - i + j|)} \prod_{\substack{i=1 \\ i \neq k}}^{\ell+1-a} \frac{Q(|r_{a+1,i} - r_{a,j} - i + j - 1|)}{Q(|r_{a+1,i} - r_{a+1,k} - i + k - 1|)}. \tag{4.15}$$

Lemma 4.4 *One has*

$$R'(\mathbf{r}, a, j) = q^{P'(\mathbf{r}, a, j)} L'(\mathbf{r}, a, j), \tag{4.16}$$

19

where

$$P'(\mathbf{r}, a, j) = \sum_{j \leq i < \ell+2-a} H_{ai}(\mathbf{r}), \tag{4.17}$$

$$L'(\mathbf{r}, a, j) = \frac{\prod_{i=1}^{\ell+1-a} Q(|r_{a+1,i} - r_{a,j} - i + j - 1|)}{\prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} Q(|r_{a,i} - r_{a,j} - i + j|)}. \tag{4.18}$$

Combining lemmas 4.3 and 4.4, we get the following expression for the CG coefficient $C_q(i, \mathbf{r}, M)$.

Lemma 4.5 *For a move $M \in \mathbb{M}_i$, let $sign\,(M)$ denote the product $\prod_{a=1}^{i-1} sign\,(m_{a+1} - m_a)$. Then one has*

$$C_q(i, \mathbf{r}, M) = sign\,(M) q^{B(M)+C(\mathbf{r},M)} \left(\prod_{a=1}^{i-1} L(\mathbf{r}, a, m_a, m_{a+1}) \right) L'(\mathbf{r}, i, m_i), \tag{4.19}$$

where

$$B(M) = \sum_{j: m_j > m_{j+1}} (2(m_j - m_{j+1} - 1) + 1), \tag{4.20}$$

$$C(\mathbf{r}, M) = \sum_{a=1}^{i-1} \left(\sum_{m_a \wedge m_{a+1} \leq b < m_a \vee m_{a+1}} H_{ab}(\mathbf{r}) + 2 \sum_{m_{a+1} < b < m_a} V_{ab}(\mathbf{r}) \right) + \sum_{m_i \leq b < \ell+2-i} H_{ib}(\mathbf{r}) \tag{4.21}$$

Lemma 4.6

$$\left(\prod_{a=1}^{i-1} L(\mathbf{r}, a, m_a, m_{a+1}) \right) L'(\mathbf{r}, i, m_i) = 1 + o(q).$$

Proof: This is a consequence of the following two inequalities:

$$|1 - (1-x)^{\frac{1}{2}}| < x \quad \text{for } 0 \leq x \leq 1,$$

and for $0 < r < 1$,

$$|1 - (1-x)^{-\frac{1}{2}}| < cx \quad \text{for } 0 \leq x \leq r,$$

where c is some fixed constant that depends on r. $\qquad\square$

Next we come to the computation of $\kappa(\mathbf{r}, \mathbf{m})$. Since $C_q(i, \mathbf{r}, \mathbf{m})$ is 0 unless $\mathbf{m}$ is of the form $M(\mathbf{r})$ for some move $M = (m_1, \ldots, m_i)$, we need only to compute $\kappa(\mathbf{r}, M(\mathbf{r}))$ which we will denote by $\kappa(\mathbf{r}, M)$.

Since

$$\psi(\mathbf{s}) = -\frac{\ell}{2} \sum_{j=1}^{\ell+1} s_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} s_{ij},$$

20

we have

$$q^{\psi(\mathbf{r})-\psi(M(\mathbf{r}))} = q^{-\frac{\ell}{2}\sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1}\sum_{j=1}^{\ell+2-i} r_{ij} + \frac{\ell}{2}(\sum_{j=1}^{\ell+1} r_{1j}+1) - (\sum_{i=2}^{\ell+1}\sum_{j=1}^{\ell+2-i} r_{ij}+i-1)} = q^{\frac{\ell}{2}-i+1}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0)$ be the top row of $\mathbf{r}$. Then

$$\min\{\psi(\mathbf{s}) : \mathbf{s}_1 = \lambda\} = -\frac{\ell}{2}\sum_1^\ell \lambda_i + \sum_{k=2}^\ell (k-1)\lambda_k.$$

Hence

$$d_\lambda = \sum_{\mathbf{s}:\mathbf{s}_1=\lambda} q^{2\psi(\mathbf{s})} = q^{-\ell\sum_1^\ell \lambda_i + 2\sum_{k=2}^\ell (k-1)\lambda_k}(1 + q^2\phi(q^2)),$$

where ϕ is a polynomial. Therefore

$$d_\lambda = q^{-\ell\sum_1^\ell \lambda_i + 2\sum_{k=2}^\ell (k-1)\lambda_k}(1 + o(q)).$$

It follows that

$$\left(\frac{d_\lambda}{d_{\lambda+e_{m_1}}}\right)^{\frac{1}{2}} = q^{\frac{\ell}{2}-m_1+1}(1 + o(q)).$$

Thus

$$\kappa(\mathbf{r}, M(\mathbf{r})) = q^{\ell+2-i-m_1}(1 + o(q)). \tag{4.22}$$

Next, observe that

$$
\begin{aligned}
& B(M) + \ell + 2 - i - m_1 \\
&= \sum_{j:m_j>m_{j+1}} (2(m_j - m_{j+1} - 1) + 1) - (m_1 - m_i) + \ell + 2 - i - m_i \\
&= 2\sum_{j:m_j>m_{j+1}} (m_j - m_{j+1}) - \sum_{j:m_j>m_{j+1}} 1 - \sum_{j=1}^{i-1}(m_j - m_{j+1}) + \ell + 2 - i - m_i \\
&= 2\sum_{j:m_j>m_{j+1}} (m_j - m_{j+1}) - \sum_{j=1}^{i-1}(m_j - m_{j+1}) - \sum_{j:m_j>m_{j+1}} 1 + \ell + 2 - i - m_i \\
&= \sum_{j:m_j>m_{j+1}} |m_j - m_{j+1}| - \#\{1 \le j \le i-1 : m_j > m_{j+1}\} + \ell + 2 - i - m_i.
\end{aligned}
$$

Thus if we write

$$A(M) = \sum_{j:m_j>m_{j+1}} |m_j - m_{j+1}| - \#\{1 \le j \le i-1 : m_j > m_{j+1}\}, \tag{4.23}$$

$$K(M) = \ell + 2 - i - m_i, \tag{4.24}$$

then both $A(M)$ and $K(M)$ are nonnegative and $B(M) + \ell + 2 - i - m_1 = A(M) + K(M)$. Thus we have

$$
\pi(u_{ij})e_{\mathbf{rs}}^{\lambda} = \sum_{\substack{M \in \mathbb{M}_i \\ M' \in \mathbb{M}_j}} C_q(i, \mathbf{r}, M(\mathbf{r}))\kappa(\mathbf{r}, M)C_q(j, \mathbf{s}, M'(\mathbf{s}))e_{M(\mathbf{r}), M'(\mathbf{s})} \tag{4.25}
$$

$$
= \sum_{\substack{M \in \mathbb{M}_i \\ M' \in \mathbb{M}_j}} \mathrm{sign}\,(M)\mathrm{sign}\,(M')q^{A(M)+K(M)+C(\mathbf{r},M)+B(M')+C(\mathbf{s},M')}(1 + o(q))e_{M(\mathbf{r}), M'(\mathbf{s})}.
$$

$$
\tag{4.26}
$$

4.2   The spectral triple

Let us briefly recall from [4] the description of the L_2 space of the sphere sitting inside $L_2(SU_q(\ell+1))$. Let $u^{\mathbb{1}}$ denote the fundamental unitary for $SU_q(\ell+1)$, i. e. the irreducible unitary representation corresponding to the Young tableaux $\mathbb{1} = (1, 0, \ldots, 0)$. Similarly write $v^{\mathbb{1}}$ for the fundamental unitary for $SU_q(\ell)$. Fix some bases for the corresponding representation spaces. Then $C(SU_q(\ell+1))$ is the C^*-algebra generated by the matrix entries $\{u_{ij}^{\mathbb{1}}\}$ and $C(SU_q(\ell))$ is the C^*-algebra generated by the matrix entries $\{v_{ij}^{\mathbb{1}}\}$. Now define ϕ by

$$
\phi(u_{ij}^{\mathbb{1}}) = \begin{cases} I & \text{if } i = j = 1, \\ v_{i-1,j-1}^{\mathbb{1}} & \text{if } 2 \le i, j \le \ell+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}
$$

Then $C(SU_q(\ell+1)\backslash SU_q(\ell))$ is the C^*-subalgebra of $C(SU_q(\ell+1))$ generated by the entries $u_{1,j}$ for $1 \le j \le \ell+1$. Define $\psi : C(S_q^{2\ell+1}) \to C(SU_q(\ell+1)\backslash SU_q(\ell))$ by

$$
\psi(z_i) = q^{-i+1}u_{1,i}^*.
$$

This gives an isomorphism between $C(SU_q(\ell+1)\backslash SU_q(\ell))$ and $C(S_q^{2\ell+1})$ and the following diagram commutes:

$$
\begin{array}{ccc}
C(S_q^{2\ell+1}) & \xrightarrow{\quad\tau\quad} & C(S_q^{2\ell+1}) \otimes C(SU_q(\ell+1)) \\
\Big\downarrow{\psi} & & \Big\downarrow{\psi \otimes \mathrm{id}} \\
C(SU_q(\ell+1)\backslash SU_q(\ell)) & \xrightarrow{\quad\Delta\quad} & C(SU_q(\ell+1)\backslash SU_q(\ell)) \otimes C(SU_q(\ell+1))
\end{array}
$$

where $\tau : C(S_q^{2\ell+1}) \to C(S_q^{2\ell+1}) \otimes C(SU_q(\ell+1))$ is the homomorphism given by $\tau(z_i) = \sum_k z_k \otimes u_{k,i}^*$ that gives an action of the quantum group $SU_q(\ell+1)$ on $S_q^{2\ell+1}$. In other words, $(C(S_q^{2\ell+1}), SU_q(\ell+1), \tau)$ is the quotient space $SU_q(\ell+1)\backslash SU_q(\ell)$. This choice of ψ makes $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ a span of certain rows of the $e_{\mathbf{r},\mathbf{s}}$'s as the following two propositions say.

22

Proposition 4.7 ([4]) *Assume $\ell > 1$. The right regular representation u of G keeps the subspace $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ invariant, and the restriction of u to $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ decomposes as a direct sum of exactly one copy of each of the irreducibles given by the young tableaux $\lambda_{n,k} := (n+k, k, k, \ldots, k, 0)$, with $n, k \in \mathbb{N}$.*

Proposition 4.8 ([4]) *Let $\mathbf{r}^{nk}$ denote the GT tableaux given by*

$$
r_{ij}^{nk} = \begin{cases} n+k & \text{if } i = j = 1, \\ 0 & \text{if } i = 1,\ j = \ell+1, \\ k & \text{otherwise,} \end{cases}
$$

where $n, k \in \mathbb{N}$. Let $\mathscr{G}_0^{n,k}$ be the set of all GT tableaux with top row $(n+k, k, \ldots, k, 0)$. Then the family of vectors

$$
\{e_{\mathbf{r}^{nk},\mathbf{s}} : n, k \in \mathbb{N},\ \mathbf{s} \in \mathscr{G}_0^{n,k}\}
$$

form a complete orthonormal basis for $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$.

We will denote $\cup_{n,k}\mathscr{G}_0^{n,k}$ by $\mathscr{G}_0$. Since the top row of $\mathbf{r}^{nk}$ determines $\mathbf{r}^{nk}$ completely and for $e_{\mathbf{r}^{nk},\mathbf{s}}$, the top row of $\mathbf{s}$ equals the top row of $\mathbf{r}^{nk}$, one can index the orthonormal basis $e_{\mathbf{r}^{nk},\mathbf{s}}$ just by $\mathbf{s} \in \mathscr{G}_0$. It was shown in [4] that the restriction of the left multiplication to $C(SU_q(\ell+1)\backslash SU_q(\ell)) \cong C(S_q^{2\ell+1})$ keeps $L_2(SU_q(\ell+1)\backslash SU_q(\ell)) \cong L_2(S_q^{2\ell+1})$ invariant. We will continue to denote this restriction by π. The operators $\pi(z_j) = q^{-j+1}\pi(u_{1,j}^*)$ will be denoted by $Z_{j,q}$. The C^*-algebra $\pi(C(S_q^{2\ell+1}))$ will be denoted by C_ℓ.

The following theorem gives a generic equivariant spectral triple for the spheres $S_q^{2\ell+1}$ constructed in [4].

Theorem 4.9 ([4]) *Let D_{eq} be the operator on $L_2(S_q^{2\ell+1})$ given by:*

$$
D_{eq}e_{\mathbf{r}^{nk},\mathbf{s}} = \begin{cases} ke_{\mathbf{r}^{nk},\mathbf{s}} & \text{if } n = 0, \\ -(n+k)e_{\mathbf{r}^{nk},\mathbf{s}} & \text{if } n > 0. \end{cases} \tag{4.28}
$$

Then $(\mathcal{A}(S_q^{2\ell+1}), L_2(S_q^{2\ell+1}), D_{eq})$ is an equivariant nondegenerate $(2\ell+1)$-summable odd spectral triple.

Our main aim in the rest of the paper is to precisely formulate the smooth function algebra for this spectral triple, establish its regularity, and compute the dimension spectrum.

4.3 The case $q = 0$

The L_2 spaces $L_2(S_q^{2\ell+1})$ for different values of q can be identified by identifying the elements of their canonical orthonormal bases which are parametrized by the same set. Thus we will

assume we are woring with one single Hilbert space $\mathcal{H}$ with orthonormal basis given by $e_{\mathbf{r}^{n,k}\mathbf{s}}$ where $\mathbf{r}^{n,k}$ is as defined earlier and $\mathbf{s}$ is given by

$$
\mathbf{s} = \begin{pmatrix}
c_1 = n+k & k & k & \cdots & k & k & d_1 = 0 \\
 & c_2 & k & k & \cdots & k & d_2 \\
 & \cdots & & & \cdots & & \\
 & c_{\ell-1} & k & d_{\ell-1} & & & \\
 & c_\ell & d_\ell & & & & \\
c_{\ell+1} = d_{\ell+1} & & & & & &
\end{pmatrix} \tag{4.29}
$$

where $c_1 \geq c_2 \geq \ldots \geq c_\ell \geq k$, $d_1 \leq d_2 \leq \ldots \leq d_\ell \leq k$ and $d_\ell \leq d_{\ell+1} \leq c_\ell$. Since specifying the GT tableaux $\mathbf{s}$ specifies $\mathbf{r}^{n,k}$ also and thus completely specifies the basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$, we will sometimes use just $\mathbf{s}$ in place of the basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$.

Let us denote by $\mathbb{M}_j^{\pm}$ the following subsets of $\mathbb{M}_j$:

$$
\begin{aligned}
\mathbb{M}_j^+ &= \{(m_1,\ldots,m_j) \in \mathbb{M}_j : m_i \in \{1, \ell+2-i\} \text{ for } 1 \leq i \leq j, \ m_1 = 1\}, \\
\mathbb{M}_j^- &= \{(m_1,\ldots,m_j) \in \mathbb{M}_j : m_i \in \{1, \ell+2-i\} \text{ for } 1 \leq i \leq j, \ m_1 = \ell+1\}.
\end{aligned}
$$

Let us denote by $N_{i,j}$ the following element of $\mathbb{M}_j$:

$$
N_{i,j} = (\underbrace{1,\ldots,1}_{i}, \ell+1-i, \ell-i, \ldots, \ell+2-j), \qquad 0 \leq i \leq j \leq \ell+1.
$$

We will denote $N_{i,\ell+1}$ by just N_i. Then from (4.26), we get

$$
\begin{aligned}
\pi(u_{1j})e_{\mathbf{r}^{n,k}\mathbf{s}} &= \sum_{M \in \mathbb{M}_j^+} \operatorname{sign}(M) q^{\ell+k+B(M)+C(\mathbf{s},M)}(1+o(q))e_{\mathbf{r}^{n+1,k},M(\mathbf{s})} \\
&\quad + \sum_{M \in \mathbb{M}_j^-} \operatorname{sign}(M) q^{B(M)+C(\mathbf{s},M)}(1+o(q))e_{\mathbf{r}^{n,k-1},M(\mathbf{s})} \tag{4.30}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Z_{j,q}^* e_{\mathbf{r}^{n,k}\mathbf{s}} &= \sum_{M \in \mathbb{M}_j^+} \operatorname{sign}(M) q^{-j+1+\ell+k+B(M)+C(\mathbf{s},M)}(1+o(q))e_{\mathbf{r}^{n+1,k},M(\mathbf{s})} \\
&\quad + \sum_{M \in \mathbb{M}_j^-} \operatorname{sign}(M) q^{-j+1+B(M)+C(\mathbf{s},M)}(1+o(q))e_{\mathbf{r}^{n,k-1},M(\mathbf{s})} \tag{4.31}
\end{aligned}
$$

Let us first look at the cases $1 \leq j \leq \ell$. In this case, the power of q in the first summation is positive. Therefore none of the terms would survive for $q = 0$. For terms in the second summation, assume $M \in \mathbb{M}_j$ with $m_1 = \ell+1$ and $m_i = 1$ for some $i \leq j$. Let $a = \min\{2 \leq i \leq j : m_i = 1\}$. Then $m_i = \ell+2-i$ for $1 \leq i \leq a-1$ so that

$$
\begin{aligned}
B(M) &\geq \sum_{i=1}^{a-2}(2((\ell+2-i)-(\ell+1-i)-1)+1) + 2(\ell+3-a-1-1)+1 \\
&= a-2+2(\ell-a+1)+1 \\
&= 2\ell-a+1.
\end{aligned}
$$

24

Hence $B(M)+1-j>0$ and so such terms will not survive for $q=0$. Thus the only term that will survive is the one corresponding to $M=N_{0,j}=(\ell+1,\ell,\ell-1,\ldots,\ell+2-j)$. In this case we have $B(M)=j-1$, $C(\mathbf{s},M)=d_j$ and $\operatorname{sign}(M)=(-1)^{j-1}$. Therefore

$$Z_{j,0}^*e_{\mathbf{r}^{n,k}\mathbf{s}}=\begin{cases}(-1)^{j-1}e_{\mathbf{r}^{n,k-1},N_{0,j}(\mathbf{s})}&\text{if }d_j=0,\\0&\text{if }d_j>0.\end{cases}\tag{4.32}$$

Next let us look at the case $j=\ell+1$. Here the first sum will be over all M with $m_1=1=m_{\ell+1}$. If $m_i\neq 1$ for some i, then $B(M)>0$ and therefore the power of q will be positive, so that the term will not survive for $q=0$. If $m_i=1$ for all i, i.e. if $M=N_\ell$, then we have $B(M)=0=C(\mathbf{s},M)$ and $\operatorname{sign}(M)=1$. Therefore for $q=0$, the first summation will become $e_{\mathbf{r}^{n+1,k},N_\ell(\mathbf{s})}$ provided $k=0$.

The second sum is over all M with $m_1=\ell+1$. Let $a=\min\{2\leq i\leq\ell+1:m_i=1\}$. Then as before, $B(M)\geq 2\ell-a+1$. Therefore if $a\leq\ell$, then $-\ell+B(M)\geq\ell-a+1>0$, so that the term will not survive for $q=0$. If $a=\ell+1$, i.e. if $M=N_0$, then $B(M)=\ell$, $C(\mathbf{s},M)=d_{\ell+1}$ and $\operatorname{sign}(M)=(-1)^\ell$. So for $q=0$, the second summation will become $(-1)^\ell e_{\mathbf{r}^{n,k-1},N_0(\mathbf{s})}$ if $k>0$ and $d_{\ell+1}=0$. Thus we have

$$Z_{\ell+1,0}^*e_{\mathbf{r}^{n,k}\mathbf{s}}=\begin{cases}e_{\mathbf{r}^{n+1,k},N_\ell(\mathbf{s})}&\text{if }k=0,\\(-1)^\ell e_{\mathbf{r}^{n,k-1},N_0(\mathbf{s})}&\text{if }k>0,\ d_{\ell+1}=0,\\0&\text{if }k>0,\ d_{\ell+1}>0.\end{cases}\tag{4.33}$$

Next we will establish a natural unitary map between $L_2(S_q^{2\ell+1})$ and

$$\mathcal{H}_\Sigma\equiv\underbrace{\ell_2(\mathbb{N})\otimes\cdots\otimes\ell_2(\mathbb{N})}_{\ell\text{ copies}}\otimes\ell_2(\mathbb{Z})\otimes\underbrace{\ell_2(\mathbb{N})\otimes\cdots\otimes\ell_2(\mathbb{N})}_{\ell\text{ copies}}.$$

For $t\in\mathbb{R}$, let t_+ denote the positive part $\max\{t,0\}$ and let t_- denote the negative part $\max\{(-t),0\}$ of t. Let us now observe that for any $\gamma\in\Gamma_\Sigma$, the tableaux

$$\mathbf{s}(\gamma):=\begin{pmatrix}\sum_1^{2\ell+1}|\gamma_i|&\sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+&\cdots&\sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+&0\\\sum_1^{2\ell}|\gamma_i|&\sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+&\cdots&&\gamma_1\\\cdots&\cdots&&&\\\sum_1^{\ell+3}|\gamma_i|&\sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+&\sum_1^{\ell-2}\gamma_i&&\\\sum_1^{\ell+2}|\gamma_i|&\sum_1^{\ell-1}\gamma_i&&&\\\sum_1^\ell\gamma_i+(\gamma_{\ell+1})_-&&&&\end{pmatrix}$$

is in $\mathscr{G}_0$. Conversely, let $\mathbf{s}\in\mathscr{G}_0^{n,k}$ for some $n,k\in\mathbb{N}$ so that $e_{\mathbf{r}^{n,k}\mathbf{s}}$ is a basis element of $L_2(S_q^{2\ell+1})$. Note that $\mathbf{s}$ is of the form (4.29). Define $\gamma\in\Gamma_\Sigma$ as follows:

1. if $k > d_{\ell+1}$, then

$$
\begin{aligned}
\gamma_i &= d_{i+1} - d_i && \text{for } 1 \le i \le \ell - 1, \\
\gamma_i &= c_{2\ell+2-i} - c_{2\ell+3-i} && \text{for } \ell + 3 \le i \le 2\ell + 1, \\
\gamma_\ell &= d_{\ell+1} - d_\ell, \quad \gamma_{\ell+1} = k - d_{\ell+1}, \quad \gamma_{\ell+2} = c_\ell - k,
\end{aligned}
$$

2. if $k \le d_{\ell+1}$, then

$$
\begin{aligned}
\gamma_i &= d_{i+1} - d_i && \text{for } 1 \le i \le \ell - 1, \\
\gamma_i &= c_{2\ell+2-i} - c_{2\ell+3-i} && \text{for } \ell + 3 \le i \le 2\ell + 1, \\
\gamma_\ell &= k - d_\ell, \quad \gamma_{\ell+1} = d_{\ell+1} - k, \quad \gamma_{\ell+2} = c_\ell - d_{\ell+1}.
\end{aligned}
$$

Then $\mathbf{s}(\gamma) = \mathbf{s}$. Thus we have a bijective correspondence between $\mathscr{G}_0$ and Γ_Σ. We will often denote a basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$ by ξ_γ using this bijective correspomdence.

Lemma 4.10 *Let $\gamma \in \Gamma_\Sigma$. For $n \in \mathbb{Z}$, let*

$$
Z_{\ell+1,0}^{(n)} := \begin{cases} Z_{\ell+1,0}^n & \text{if } n \ge 0, \\ (Z_{\ell+1,0}^*)^{-n} & \text{if } n < 0. \end{cases}
$$

Define

$$
\xi'_\gamma := Z_{1,0}^{\gamma_1} \ldots Z_{\ell,0}^{\gamma_\ell} Z_{\ell+1,0}^{(\gamma_{\ell+1})} \begin{pmatrix} \sum_{\ell+2}^{2\ell+1} \gamma_i & 0 & \cdots & 0 & 0 \\ \sum_{\ell+2}^{2\ell} \gamma_i & 0 & \cdots & 0 \\ & & \cdots & \\ \gamma_{\ell+2} & 0 & & \\ 0 & & & \end{pmatrix}.
$$

Then $\{\xi'_\gamma : \gamma \in \Gamma_\Sigma\}$ is an orthonormal basis for $L_2(S_q^{2\ell+1})$.

Proof: It follows from equations (4.32) and (4.33) that the actions of $Z_{j,0}$ for $1 \le j \le \ell$ on the

basis elements $e_{\mathbf{r}^{n,k}\mathbf{s}}$ are as follows:

$$
Z_{j,0} : \begin{pmatrix}
n+k & k & & \cdots & & k & k & 0 \\
& c_2 & k & & \cdots & & k & 0 \\
& \cdots & & \cdots & & & & \\
& c_j & k & \cdots & & k & 0 & \\
& c_{j+1} & k & \cdots & d_{j+1} & & & \\
& \cdots & & \cdots & & & & \\
& c_\ell & d_\ell & & & & & \\
& d_{\ell+1} & & & & & &
\end{pmatrix} \longrightarrow
$$

$$
(-1)^{j-1} \begin{pmatrix}
1+n+k & 1+k & & \cdots & & 1+k & 1+k & 0 \\
& 1+c_2 & 1+k & & \cdots & & 1+k & 0 \\
& \cdots & & \cdots & & & & \\
& 1+c_j & 1+k & \cdots & & 1+k & 0 & \\
& 1+c_{j+1} & 1+k & \cdots & 1+d_{j+1} & & & \\
& \cdots & & \cdots & & & & \\
& 1+c_\ell & 1+d_\ell & & & & & \\
& 1+d_{\ell+1} & & & & & &
\end{pmatrix}
$$

and is 0 for $\mathbf{s}$ with $d_j > 0$.

Similarly the action of $Z_{\ell+1,0}$ on the basis elements are as follows:

$$
\begin{pmatrix}
n & 0 & & \cdots & & 0 & 0 \\
c_2 & 0 & & \cdots & & 0 & \\
\cdots & & \cdots & & & & \\
c_{\ell-1} & 0 & 0 & & & & \\
c_\ell & 0 & & & & & \\
d_{\ell+1} & & & & & &
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
n-1 & 0 & & \cdots & & 0 & 0 \\
c_2-1 & 0 & & \cdots & & 0 & \\
\cdots & & \cdots & & & & \\
c_{\ell-1}-1 & 0 & 0 & & & & \\
c_\ell-1 & 0 & & & & & \\
d_{\ell+1}-1 & & & & & &
\end{pmatrix}
$$

if $d_{\ell+1} > 0$, and

$$
\begin{pmatrix}
n+k & k & & \cdots & & k & 0 \\
c_2 & k & & \cdots & & 0 & \\
\cdots & & \cdots & & & & \\
c_{\ell-1} & k & 0 & & & & \\
c_\ell & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}
\longrightarrow (-1)^\ell \begin{pmatrix}
1+n+k & 1+k & & \cdots & & 1+k & 0 \\
1+c_2 & 1+k & & \cdots & & 0 & \\
\cdots & & \cdots & & & & \\
1+c_{\ell-1} & 1+k & 0 & & & & \\
1+c_\ell & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}
$$

if $d_{\ell+1} = 0$. Similarly the action of $Z^*_{\ell+1,0}$ on the basis elements are as follows:

$$
\begin{pmatrix}
n & 0 & \cdots & 0 & 0 \\
c_2 & 0 & \cdots & 0 & \\
\cdots & & \cdots & & \\
c_{\ell-1} & 0 & 0 & & \\
c_\ell & 0 & & & \\
d_{\ell+1} & & & &
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1+n & 0 & \cdots & 0 & 0 \\
1+c_2 & 0 & \cdots & 0 & \\
\cdots & & \cdots & & \\
1+c_{\ell-1} & 0 & 0 & & \\
1+c_\ell & 0 & & & \\
1+d_{\ell+1} & & & &
\end{pmatrix}
$$

and for $k > 0$,

$$
\begin{pmatrix}
n+k & k & \cdots & k & 0 \\
c_2 & k & \cdots & 0 & \\
\cdots & & \cdots & & \\
c_{\ell-1} & k & 0 & & \\
c_\ell & 0 & & & \\
0 & & & &
\end{pmatrix}
\longrightarrow (-1)^\ell
\begin{pmatrix}
n+k-1 & k-1 & \cdots & k-1 & 0 \\
c_2-1 & k-1 & \cdots & 0 & \\
\cdots & & \cdots & & \\
c_{\ell-1}-1 & k-1 & 0 & & \\
c_\ell-1 & 0 & & & \\
0 & & & &
\end{pmatrix}
$$

Then it follows from the above that

$$
Z^{\gamma_1}_{1,0}\ldots Z^{\gamma_\ell}_{\ell,0} Z^{(\gamma_{\ell+1})}_{\ell+1,0}
\begin{pmatrix}
\sum_{\ell+2}^{2\ell+1}\gamma_i & 0 & \cdots & 0 & 0 \\
\sum_{\ell+2}^{2\ell}\gamma_i & 0 & \cdots & 0 & \\
& & \cdots & & \\
\gamma_{\ell+2} & 0 & & & \\
0 & & & &
\end{pmatrix}
$$

$$
= (-1)^{\eta(\gamma)}
\begin{pmatrix}
\sum_1^{2\ell+1}|\gamma_i| & \sum_1^\ell \gamma_i+(\gamma_{\ell+1})_+ & \cdots & \sum_1^\ell \gamma_i+(\gamma_{\ell+1})_+ & 0 \\
\sum_1^{2\ell}|\gamma_i| & \sum_1^\ell \gamma_i+(\gamma_{\ell+1})_+ & \cdots & & \gamma_1 \\
\cdots & \cdots & & & \\
\sum_1^{\ell+3}|\gamma_i| & \sum_1^\ell \gamma_i+(\gamma_{\ell+1})_+ & \sum_1^{\ell-2}\gamma_i & & \\
\sum_1^{\ell+2}|\gamma_i| & \sum_1^{\ell-1}\gamma_i & & & \\
\sum_1^\ell \gamma_i+(\gamma_{\ell+1})_- & & & &
\end{pmatrix},
$$
(4.34)

where $\eta(\gamma) := \sum_{i=1}^\ell (i-1)\gamma_i + \ell(\gamma_{\ell+1})_+$. Thus $\xi'_\gamma = (-1)^{\eta(\gamma)}\xi_\gamma$. Therefore it follows that $\{\xi'_\gamma : \gamma \in \Gamma_\Sigma\}$ is an orthonormal basis for $L_2(S_q^{2\ell+1})$. $\qquad\square$

The map $U : L_2(S_q^{2\ell+1}) \to \mathcal{H}_\Sigma$ given by $U\xi'_\gamma = e_\gamma$ sets up a unitary isomorphism between $L_2(S_q^{2\ell+1})$ and $\mathcal{H}_\Sigma$. Let P denote the projection onto the span of $e_0 \otimes \cdots \otimes e_0$ in $\ell_2(\mathbb{N}^\ell)$. Then we have

$$
U Z_{j,0} U^* = Y_{j,0} \otimes I = Y_{j,0} \otimes P + Y_{j,0} \otimes (I-P),
$$
(4.35)

28

and

$$UD_{eq}U^* = D_\ell \otimes P - |D_\ell| \otimes (I - P) - I \otimes \widetilde{N}, \qquad (4.36)$$

where $\widetilde{N}$ is the operator $e_{m_1} \otimes \cdots \otimes e_{m_\ell} \mapsto \left(\sum m_i\right) e_{m_1} \otimes \cdots \otimes e_{m_\ell}$. In other words, with respect to the decomposition

$$\mathcal{H}_\Sigma = \mathcal{H}_\ell \oplus \left(\mathcal{H}_\ell \otimes \ell_2(\mathbb{N}^\ell\backslash\{0,\ldots,0\})\right),$$

one has

$$UZ_{j,0}U^* = Y_{j,0} \oplus (Y_{j,0} \otimes I),$$

and

$$UD_{eq}U^* = D_\ell \oplus \left(-|D_\ell| \otimes I - I \otimes \widetilde{N}\right).$$

Next we will define the smooth function algebra $C_{eq}^\infty(S_0^{2\ell+1})$ and prove that the spectral triple $(C_{eq}^\infty(S_0^{2\ell+1}), \mathcal{H}, D_{eq})$ is regular with simple dimension spectrum $\{1, 2, \ldots, 2\ell+1\}$.

It follows from decomposition (4.35) that if we identify $L_2(S_q^{2\ell+1})$ with $\mathcal{H}_\Sigma$, then the C^*-algebra generated by the $Z_{j,0}$'s is $A_\ell \otimes I$, where A_ℓ is the C^*-algebra generated by the Y_j's in $\mathcal{L}(\mathcal{H}_\ell)$. Therefore it is natural to define

$$C_{eq}^\infty(S_0^{2\ell+1}) = \{a \otimes I : a \in \mathcal{A}_\ell^\infty\}. \qquad (4.37)$$

Theorem 4.11 *The triple* $(C_{eq}^\infty(S_0^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ *is a regular spectral triple with simple dimension spectrum* $\{1, 2, \ldots, 2\ell+1\}$.

Proof: Since $\mathcal{A}_\ell^\infty$ is closed under holomorphic function calculus in A_ℓ, it follows that $C_{eq}^\infty(S_0^{2\ell+1})$ is closed under holomorphic function calculus in $C^*(\{Z_{j,0} : 1 \le j \le \ell+1\}) = A_\ell \otimes I$. In order to show regularity, let us introduce the algebra

$$\mathcal{B}_{eq} := \{a \otimes P + b \otimes (I - P) : a, b \in \mathcal{B}_\ell\} \qquad (4.38)$$

Clearly $\mathcal{B}_{eq}$ contains $C_{eq}^\infty(S_0^{2\ell+1})$. We will show that $\mathcal{B}_{eq}$ is closed under derivations with both $|D_{eq}|$ as well as D_{eq}. This will prove regularity of the spectral triple $(C_{eq}^\infty(S_0^{2\ell+1}), \mathcal{H}, D_{eq})$.

Note that $|D_{eq}| = |D_\ell| \otimes I + I \otimes \widetilde{N}$. Since $I \otimes \widetilde{N}$ commutes with every element of $\mathcal{B}_{eq}$, we get $\delta(a \otimes P + b \otimes (I - P)) = [|D_\ell|, a] \otimes P + [|D_\ell|, b] \otimes (I - P)$ and $[D_{eq}, a \otimes P + b \otimes (I - P)] = [D_\ell, a] \otimes P - [|D_\ell|, b] \otimes (I - P)$. Since $\mathcal{B}_\ell$ is closed under derivations with $|D_\ell|$ and D_ℓ, it follows that $\mathcal{B}_{eq}$ is closed under derivations with $|D_{eq}|$ and D_{eq}.

Next we compute the dimension spectrum of the spectral triple. For $w \in \mathbb{T}^{\ell+1}$, let $\tilde{U}_w := U_w \otimes I$. Then $|D_{eq}|$ commutes with $\tilde{U}_w$. Hence again it is enough to consider homogeneous elements of degree 0. Now by lemma 3.7 it follows that for $b \in \mathcal{B}_{eq}$ with b homogeneous of degree 0, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, 2\ell+1\}$. To show that every point of $\{1, 2, \cdots, 2\ell+1\}$ is in the dimension spectrum, observe that

$$Trace(|D_{eq}|^{-z}) = \Sigma_{k=0}^{2\ell}(2c_k^{2\ell} - c_k^{2\ell-1})\zeta(z - k) \qquad (4.39)$$

where c_k^r is defined as the coefficient of N^k in $\binom{N+r}{r}$. Note that for $0 \leq k \leq r$ one has $c_k^r > 0$. Also note the recurrence $rc_k^r = c_{k-1}^{r-1} + rc_k^{r-1}$. Hence $c_k^r \geq c_k^{r-1}$. Now from equation (4.39) it follows that $Res_{z=k+1}Trace(|D_{eq}|^{-z}) = 2c_k^{2\ell} - c_k^{2\ell-1} > 0$ for $0 \leq k \leq 2\ell$. This proves that every point of $\{1, 2, \cdots, 2\ell + 1\}$ is in the dimension spectrum. This completes the proof. $\square$

We will need the fact that $Trace(|D_{eq}|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, 2\ell + 1\}$ with non-zero residue and hence we state it as a separate lemma.

Lemma 4.12 *The function $Trace(|D_{eq}|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, 2\ell + 1\}$. Also for $k \in \{1, 2, \cdots, 2\ell + 1\}$, the residue $Res_{z=k}Trace(|D_{eq}|^{-z})$ is non-zero.*

4.4 Regularity and dimension spectrum for $q \neq 0$

Consider the smooth subalgebra of the Toeplitz algebra defined as:

$$\mathscr{T}^\infty \;\; = \;\; \left\{ \sum_{j,k \in \mathbb{N}} \lambda_{jk} \, S^{*j}p_0 S^k + \sum_{k \geq 0} \lambda_k S^k + \sum_{k > 0} \lambda_{-k} S^{*k} : \lambda_{jk}, (\lambda_k) \text{ are rapidly decreasing} \right\}$$

For $a := \sum_{j,k \in \mathbb{N}} \lambda_{jk} \, S^{*j}p_0 S^k + \sum_{k \geq 0} \lambda_k S^k + \sum_{k > 0} \lambda_{-k} S^{*k} \in \mathscr{T}^\infty$, define the seminorm $\| \cdot \|_m$ by $\|a\|_m := \sum(1 + |j| + |k|)^m |\lambda_{kl}| + \sum(1 + |k|)^m |\lambda_k|$. Equipped with this family of seminorms, $\mathscr{T}^\infty$ is a Fréchet algebra. We will denote by $\mathscr{T}_k^\infty$ the k-fold tensor product of $\mathscr{T}^\infty$.

Lemma 4.13 *The triple $(\mathscr{T}^\infty, \ell_2(\mathbb{N}), N)$ is a regular spectral triple. More precisely, $\mathscr{T}^\infty$ is contained in $Dom(\delta)$ where δ is the unbounded derivation $[N, .]$ and δ leaves the algebra $\mathscr{T}^\infty$ invariant. Also the map $\delta : \mathscr{T}^\infty \to \mathscr{T}^\infty$ is continuous.*

Proof: Note that $[N, S] = -S$ and $[N, p] = 0$. Now the lemma follows from the fact that the unbounded derivation δ is closed. $\square$

For $\alpha \in \mathbb{N}^2 \cup \mathbb{Z}$, let

$$W_\alpha \;\; = \;\; \begin{cases} S^{*m}p_0 S^n & \text{if} \quad \alpha = (m, n) \\ S^r & \text{if} \quad \alpha = r \geq 0 \\ S^{*r} & \text{if} \quad \alpha = r < 0 \end{cases}$$

For $\alpha \in \mathbb{N}^2 \cup \mathbb{Z}$, define $|\alpha|$ to be $|m| + |n|$ if $\alpha = (m, n) \in \mathbb{N}^2$ and the usual absolute value $|\alpha|$ if $\alpha \in \mathbb{Z}$. For an ℓ tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell)$ in $(\mathbb{N}^2 \cup \mathbb{Z})^\ell$, let $|\alpha| = \sum |\alpha_i|$ and $W_\alpha := W_{\alpha_1} \otimes W_{\alpha_2} \otimes \cdots W_{\alpha_\ell}$. We need the following simple lemma whose proof we omit as it is easy to prove.

Lemma 4.14 *The natural tensor product representation of $\mathscr{T}_\ell^\infty$ on $\ell_2(\mathbb{N})^{\otimes \ell}$ is injective. Thus we identify $\mathscr{T}_\ell^\infty$ with it's range which is $\{\sum x_\alpha W_\alpha : \sum(1 + |\alpha|)^p |x_\alpha| < \infty$ for every $p\}$.*

Remark 4.15 The tensor product representation of $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ on $\mathcal{L}(\mathcal{H}_\Sigma)$ is injective since $OP_{D_\ell}^{-\infty} := \mathcal{S}(\mathcal{H}_\ell)$ and hence we identify $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ with its image.

For an operator T, let L_T denote the left multiplication map $X \mapsto TX$. Then for $T \in OP_{D_\ell}^0$, the map $L_T : OP_{D_\ell}^{-\infty} \to OP_{D_\ell}^{-\infty}$ is continuous. Note that if A is a Fréchet algebra and $a \in A$, then L_a is a continuous linear operator.

Lemma 4.16 *Let $T \in OP_{D_\ell}^0$ and $a \in \mathscr{T}_\ell^\infty$. Then the map $L_{T \otimes a}$ leaves the algebra $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ invariant. Moreover $L_{T \otimes a} = L_T \otimes L_a$ on the algebra $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$.*

Proof: Clearly $L_{T \otimes a} = L_T \otimes L_a$ on the algebraic tensor product $OP_{D_\ell}^{-\infty} \otimes_{alg} \mathscr{T}_\ell^\infty$. Now let $a \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Then there exists a sequence $a_n \in OP_{D_\ell}^{-\infty} \otimes_{alg} \mathscr{T}_\ell^\infty$ which converges to a in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Also a_n converges to a in the operator norm. Now the result follows from the continuity of $L_{T \otimes a}$ and $L_T \otimes L_a$. $\qquad\qquad\square$

Proposition 4.17 *Let*

$$\mathscr{B} := \mathcal{B}_{eq} + OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty \qquad\qquad (4.40)$$

Then one has the following.

1. *The vector space $\mathscr{B}$ is an algebra.*

2. *The algebra $\mathscr{B}$ is invariant under the derivations $\delta := [|D_{eq}|, .]$ and $[D_{eq}, .]$.*

3. *For $b \in \mathscr{B}$, the commutator $[F_{eq}, b] \in OP_{D_{eq}}^{-\infty}$.*

4. *For $b \in \mathscr{B}$, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with only simple poles and the poles lie in $\{1, 2, \cdots, 2\ell + 1\}$.*

Proof: Lemma 4.16 and the fact that $\mathcal{B}_\ell \subset OP^0$ implies that $\mathscr{B}$ is an algebra. As seen in Theorem 4.11, it follows that $\mathcal{B}_{eq}$ is invariant under δ and $[D_{eq}, .]$. Also (3) and (4) holds for $b \in \mathcal{B}_{eq}$. Hence to complete the proof it is enough to consider $(2), (3)$ and (4) for the algebra $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$.

Lemma 2.6 and the decomposition $|D_{eq}| = |D_\ell| \otimes 1 + 1 \otimes \tilde{N}$ implies that δ leaves the algebra $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ invariant. Now note that $P \in OP_{\tilde{N}}^{-\infty}$, it follows that left and right multiplication by $F_\ell \otimes P$ and $1 \otimes P$ sends $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ to $OP_{D_{eq}}^{-\infty} \equiv OP_{D_\ell}^{-\infty} \otimes OP_{\tilde{N}}^{-\infty}$. Now since $F_{eq} = F_\ell \otimes P - I \otimes (I - P)$, it follows that $[F_{eq}, b]$ is smoothing for every $b \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}^\infty$. Now the invariance of $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ under $[D_{eq}, .]$ follows from the equation $[D_{eq}, b] = \delta(b) F_{eq} + |D_{eq}|[F_{eq}, b]$ and the fact that $OP_{D_{eq}}^{-\infty} := OP_{D_\ell}^{-\infty} \otimes OP_{\tilde{N}}^{-\infty}$ is contained in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$.

Now we will prove that for $b \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, \ell\}$. For $w \in \mathbb{T}^{2\ell+1}$, let $U_w = U_{w_1} \otimes U_{w_2} \otimes \cdots U_{w_{2\ell+1}}$ be the unitary operator on $\mathcal{H}_\Sigma$. Clearly $U_w |D_{eq}| U_w^* = |D_{eq}|$ for $w \in \mathbb{T}^{2\ell+1}$. Hence it is enough to consider $Trace(b|D_{eq}|^{-z})$ with b homogeneous of degree 0.

An element b is homogeneous if and only if it commutes with the operators U_w for all $w \in \mathbb{T}^{2\ell+1}$. This implies b must be of the form $e_\gamma \mapsto \phi(\gamma)e_\gamma$ for some function ϕ, i.e. $b = \sum_\gamma \phi(\gamma)p_\gamma$. An operator of the form $\sum_{\gamma \in \Gamma_{\Sigma_\ell}} \phi(\gamma)p_\gamma$ is in $OP_{D_\ell}^{-\infty}$ if and only if $\phi(\gamma)$ is rapidly decaying on Γ_{Σ_ℓ}. Also, using the description of $\mathscr{T}^\infty$, it follows that an operator of the form $\sum_{n \in \mathbb{N}} \phi(n)p_n$ belongs to $\mathscr{T}^\infty$ if and only if $\phi(\cdot) - \lim_{n \to \infty} \phi(n)$ is rapidly decreasing. Thus combining these, one can see that the operator $\sum_\gamma \phi(\gamma)p_\gamma$ belongs to $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ if and only if ϕ is a linear combination of ϕ_A with A varying over subsets of Σ containing Σ_ℓ, where each $\phi_A(\gamma)$ depends only on γ_A and $\phi_A(\gamma_A)$ is rapidly decreasing on Γ_A. For an element $b = \sum_\gamma \phi_A(\gamma)p_\gamma$, one has

$$Trace(b|D_{eq}|^{-z}) = \sum_\gamma \frac{\phi_A(\gamma)}{|\gamma|^z} = \sum_\gamma \frac{\phi_A(\gamma_A)}{(|\gamma_A| + |\gamma_{\Sigma \setminus A}|)^z}.$$

By lemma 3.7 it follows that $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, |\Sigma \setminus A|\} \subseteq \{1, 2, \ldots, \ell\}$. This completes the proof. $\qquad \square$

4.5   The smooth function algebra $C^\infty(S_q^{2\ell+1})$

In this subsection we will define a dense $*$ Fréchet algebra $C^\infty(S_q^{2\ell+1})$ of $C_\ell = \pi(C(S_q^{2\ell+1}))$ and show that it is closed under holomorphic functional calculus. Let B_ℓ be the C^* algebra generated by A_ℓ and F_ℓ. Note that B_ℓ contains $\mathcal{K}(\ell_2(\mathbb{N})^{\otimes \ell} \otimes \ell_2(\mathbb{Z}))$. Recall that $\mathcal{E}$ denotes the C^* algebra generated by $C(\mathbb{T})$ and F_0.

Lemma 4.18 *The C^* algebra $\mathcal{E}$ contains $\mathcal{K}$ and $\mathcal{E}/\mathcal{K}$ is isomorphic to the C^* algebra $C(\mathbb{T}) \oplus C(\mathbb{T})$.*

Proof: Let $|e_m\rangle\langle e_n|$ be the matrix units in $\mathcal{K}(\ell_2(\mathbb{Z}))$. Note that $[F_0, S^*]S = 2|e_0\rangle\langle e_0|$. Hence $p_0 \equiv |e_0\rangle\langle e_0| \in \mathcal{E}$. Now $S^{*m}p_0S^n = |e_m\rangle\langle e_n|$. Hence $\mathcal{K} \subset \mathcal{E}$. Let $P_0 := \frac{1+F_0}{2}$. Then $[P_0, f]$ is compact for every f. Thus $\mathcal{E}/\mathcal{K}$ is generated by $C(\mathbb{T})$ and a projection P_0 which is in the center of $\mathcal{E}/\mathcal{K}$. Now consider the map

$$C(\mathbb{T}) \oplus C(\mathbb{T}) \ni (f, g) \mapsto fP_0 + g(1 - P_0) \quad (mod \ \mathcal{K}) \in \mathcal{E}/\mathcal{K}.$$

We claim that this map is an isomorphism. To prove we need to show that if fP_0 is compact then $f = 0$ and if $g(1 - P_0)$ is compact then $g = 0$.

Assume that fP_0 is compact for $f \in C(\mathbb{T})$. Fix an $r \in \mathbb{Z}$. Since fP_0 is compact, it follows that $|\langle fP_0(e_n), e_{n+r}\rangle| = |\hat{f}(r)|$ converges to 0 as $n \to +\infty$. Hence $\hat{f}(r) = 0$ for every r. This proves that $f = 0$. Similarly one can show that if $g(1 - P_0)$ is compact then $g = 0$. This completes the proof. $\qquad \square$

Lemma 4.19 *The C^*-algebra B_ℓ contains $\mathcal{K}(\mathcal{H}_\ell)$ and the map $(a, b) \mapsto aP_\ell + b(1 - P_\ell) \mod \mathcal{K}$ from $C(S_q^{2\ell+1}) \oplus C(S_q^{2\ell+1})$ to $B_\ell/\mathcal{K}(\mathcal{H}_\ell)$ is an isomorphism.*

Proof: For $\ell = 0$ this is just lemma 4.18. So let us prove the statement for $\ell \geq 1$. Since A_ℓ contains $\mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T})$, it follows that B_ℓ contains $\mathcal{K}(\mathcal{H}_\ell)$. Observe that $[P_\ell, \alpha_i] = 0$ for $1 \leq i \leq \ell$ and $[P_\ell, \alpha_{\ell+1}]$ is compact. Therefore it follows that $[P_\ell, a]$ is compact for every $a \in A_\ell$. Hence the map $(a, b) \mapsto aP_\ell + b(1 - P_\ell) \mod \mathcal{K}$ from $A_\ell \oplus A_\ell$ to $B_\ell/\mathcal{K}$ is a $*$ algebra homomorphism onto $B_\ell/\mathcal{K}$. We will show that the map is one-one. For that we have to show if aP_ℓ is compact with $a \in A_\ell$ then $a = 0$ and if $b(1 - P_\ell)$ is compact with $b \in A_\ell$ then $b = 0$.

Suppose now that aP_ℓ is compact. Observe that $B_\ell \subset \mathcal{T}_\ell \otimes \mathcal{E}$ and $aP_\ell = a(I \otimes P_0)$. Since $aP_\ell = 0$, if we apply the symbol map σ on the ℓth copy of $\mathcal{T}$, we get $\sigma_\ell(a) \otimes P_0 = 0$. Hence a is in the ideal $\mathcal{K}(\ell_2(\mathbb{N})^{\otimes \ell}) \otimes C(\mathbb{T})$. For $m, n \in \mathbb{N}^\ell$, let e_{mn} be the "matrix" units. Let $a_{mn} = (e_{mm} \otimes 1)a(e_{nn} \otimes 1)$. Then $a_{mn} = e_{mn} \otimes f_{mn}$ for some $f_{mn} \in C(\mathbb{T})$. Since aP_ℓ is compact, it follows that $f_{mn}P_0$ is compact as $P_\ell = I \otimes P_0$ commutes with $e_{nn} \otimes I$. By the $\ell = 0$ case, it follows that $f_{mn} = 0$ and hence $a_{mn} = 0$ for every m, n. Thus $a = 0$. Similarly one can show that if $b(1 - P_\ell)$ is compact then $b = 0$. This completes the proof. $\square$

Let $\mathbb{B}$ be the C^* algebra on $\mathcal{H}_\Sigma$ generated by $A_\ell \otimes I$, $P_\ell \otimes 1$ and $1 \otimes P$ and $J := \mathcal{K}(\mathcal{H}_\ell) \otimes \mathcal{T}_\ell$. Note that J is an ideal since $\mathbb{B}_\ell$ is contained in $\mathcal{T}_\ell \otimes \mathcal{E} \otimes \mathcal{T}_\ell$. The next proposition identifies the quotient $\mathbb{B}/J$.

Proposition 4.20 *Let* $\rho : A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell \to \mathbb{B}/J$ *be the map*

$$(a_1, a_2, a_3, a_4) \mapsto a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1 - P) + a_3 (1 - P_\ell) \otimes P + a_4 (1 - P_\ell) \otimes (1 - P)$$

from $A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell$ *into* $\mathbb{B}$ *composed with the canonical projection from* $\mathbb{B}$ *onto* $\mathbb{B}/J$. *Then* ρ *is an isomorphism.*

Proof: First note that since $[P_\ell, a] \in \mathcal{K}$ for $a \in A_\ell$, it follows that $P_\ell \otimes I$ and $I \otimes P$ are in the center of $\mathbb{B}/J$. Hence the map ρ is an algebra homomorphism. By the definition of $\mathbb{B}$ it follows that ρ is onto. Thus we have to show ρ is one-one.

Suppose that $a = a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1 - P) + a_3 (1 - P_\ell) \otimes P + a_4 (1 - P_\ell) \otimes (1 - P) \in J$. Let $\epsilon : \mathcal{T} \to \mathbb{C}$ be the map $ev_1 \circ \sigma$, where ev_1 is evaluation at the point 1. Now consider the map $id \otimes \epsilon^{\otimes \ell} : \mathcal{T}_\ell \otimes \mathcal{E} \otimes \mathcal{T}_\ell \to \mathcal{T}_\ell \otimes \mathcal{E}$. Note that $I \otimes \epsilon^{\otimes \ell}$ sends J to $\mathcal{K}(\mathcal{H}_\ell)$. Hence $(I \otimes \epsilon^{\otimes \ell})(a) = a_2 P_\ell + a_4(1 - P_\ell) \in \mathcal{K}(\mathcal{H}_\ell)$. Hence by lemma 4.19, it follows that $a_2 = 0 = a_4$. Since left multiplication by $I \otimes P$ sends the ideal J to $\mathcal{K}(\mathcal{H}_\Sigma)$. It follows that $(I \otimes P)a = a_1 P_\ell \otimes P + a_3(1 - P_\ell) \otimes P$ is compact. Hence $a_1 P_\ell + a_3(1 - P_\ell)$ is compact. Thus again by lemma 4.19, it follows that $a_1 = 0 = a_3$. This completes the proof. $\square$

Now we prove that $\mathscr{B}$ is closed under holomorphic functional calculus in $\mathbb{B}$. Let $\mathscr{J} := OP_{D_\ell}^{-\infty} \otimes \mathcal{T}_\ell^\infty$. Note that

$$\mathscr{B} := \{a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1 - P) + a_3 (1 - P_\ell) \otimes P + a_4 (1 - P_\ell) \otimes (1 - P) + R :$$
$$a_1, a_2, a_3, a_4 \in A_\ell^\infty, R \in \mathscr{J}\}$$

Proposition 4.21 *The algebra $\mathscr{B}$ has the following properties:*

1. *If $a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1 - P) + a_3(1 - P_\ell) \otimes P + a_4(1 - P_\ell) \otimes (1 - P) \in \mathscr{J}$ then $a_i = 0$ for $i = 1, 2, 3, 4$. Hence $\mathscr{B}$ is isomorphic to the direct sum $A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus \mathscr{J}$. Equip $\mathscr{B}$ with the Fréchet space structure coming from this direct sum decomposition.*

2. *The algebra $\mathscr{B}$ is a $*$-Fréchet algebra contained in $\mathbb{B}$. Moreover the inclusion $\mathscr{B} \subset \mathbb{B}$ is continuous.*

3. *The algebra $\mathscr{B}$ is closed under holomorphic functional calculus in $\mathbb{B}$.*

Proof: Proposition 4.20 implies (1). Parts (2) and (3) follows from proposition 3.6. Now by proposition 4.20 one has the exact sequence

$$0 \to J \to \mathbb{B} \to A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell \to 0.$$

At the smooth algebra level we have the following exact sequence

$$0 \to \mathscr{J} \to \mathscr{B} \xrightarrow{\theta} A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \to 0.$$

Since $\mathscr{J} \subset J$ and $A_\ell^\infty \subset A_\ell$ are closed under holomorphic functional calculus, it follows from theorem 3.2, part 2, [13] that $\mathscr{B}$ is spectrally invariant in $\mathbb{B}$. Since by part (2), the Fréchet topology of $\mathscr{B}$ is finer than the norm topology, it follows that $\mathscr{B}$ is closed in the holomorphic function calculus of $\mathbb{B}$. $\qquad\square$

Remark 4.22 One can prove that $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ is closed under holomorphic functional calculus in $\mathcal{K}(\mathcal{H}_\ell) \otimes \mathscr{T}_\ell$ in the same manner by applying theorem 3.2, part 2, [13] and by using the extension (after tensoring suitably)

$$0 \to \mathcal{K} \to \mathscr{T} \to C(\mathbb{T}) \to 0$$

at the C^* algebra level and the extension

$$0 \to \mathcal{S}(\ell_2(\mathbb{N})) \to \mathscr{T}^\infty \to C^\infty(\mathbb{T}) \to 0$$

at the Fréchet algebra level.

Corollary 4.23 *Define the smooth function algebra $C^\infty(S_q^{2\ell+1})$ by*

$$C^\infty(S_q^{2\ell+1}) = \{a \in \mathscr{B} \cap C_\ell : \theta(a) \in \iota(\mathcal{A}_\ell^\infty)\},$$

where θ is as in the proof of proposition 4.21 and $\iota : A_\ell \to A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell$ is the inclusion map $a \mapsto a \oplus a \oplus a \oplus a$. Then the algebra $C^\infty(S_q^{2\ell+1})$ is closed in $\mathscr{B}$ and it is closed under holomorphic functional calculus in C_ℓ.

34

Proof: Let $j : \mathscr{B} \to \mathcal{L}(\mathcal{H}_\Sigma)$ denote the inclusion map. Then by definition $C^\infty(S_q^{2\ell+1}) = \theta^{-1}(\iota(A_\ell^\infty)) \cap j^{-1}(C_\ell)$. Since θ and j are continuous and as $\iota(A_\ell^\infty)$ and C_ℓ are closed, it follows that $C^\infty(S_q^{2\ell+1})$ is closed in $\mathscr{B}$. Hence $C^\infty(S_q^{2\ell+1})$ is a Fréchet algebra. Also $C^\infty(S_q^{2\ell+1})$ is $*$-closed as ρ is $*$-preserving. Now let $a \in C^\infty(S_q^{2\ell+1})$ be invertible in C_ℓ. Then a is invertible in $\mathcal{L}(\mathcal{H}_\Sigma)$. By proposition 4.21, it follows that $a^{-1} \in \mathscr{B}$. By the closedness of A_ℓ^∞ under holomorphic functional calculus, it follows that $\theta(a^{-1}) \in \iota(A_\ell^\infty)$. Thus one has $a^{-1} \in C^\infty(S_q^{2\ell+1})$. We have already seen that the Fréchet topology of $\mathscr{B}$ is finer than the norm topology. Same is therefore true for the topology of $C^\infty(S_q^{2\ell+1})$. Hence it is closed under holomorphic functional calculus in C_ℓ. $\qquad\square$

Proposition 4.24 *The operators $Z_{j,q} \in C^\infty(S_q^{2\ell+1})$. Hence $C^\infty(S_q^{2\ell+1})$ is a dense subalgebra of C_ℓ that contains $\pi(\mathcal{A}(S_q^{2\ell+1}))$.*

The proof of this proposition will be given in the next subsection.

We are now in a position to prove the main theorem.

Theorem 4.25 *The triple $(C^\infty(S_q^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ is a regular spectral triple with simple dimension spectrum $\{1, 2, \cdots, 2\ell + 1\}$.*

Proof: Since $C^\infty(S_q^{2\ell+1}) \subset \mathscr{B}$ the regularity of the spectral triple $(C^\infty(S_q^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ follows from the regularity of the spectral triple $(\mathscr{B}, \mathcal{H}_\Sigma, D_{eq})$ which is proved in proposition 4.17. Proposition 4.17 also implies that the spectral triple has simple dimension spectrum which is a subset of $\{1, 2, \cdots, 2\ell + 1\}$. The fact that every point in $\{1, 2, \cdots, 2\ell + 1\}$ is in the dimension spectrum follows from lemma 4.12. This completes the proof. $\qquad\square$

4.6 The operators $Z_{j,q}$

We will give a proof of proposition 4.24 in this subsection. The main idea will be to exploit the isomorphism between the Hilbert spaces $L_2(S_q^{2\ell+1})$ and $\mathcal{H}_\Sigma$ and a detailed analysis of the operators $Z_{j,q}$ to show that certain parts of these operators can be ignored for the purpose of establishing regularity and computing dimension spectrum. Deciding and establishing which parts of these operators can be ignored is the key step here. It should be noted here that a similar analysis has been done by D'Andrea in [9], where he embeds $L_2(S_q^{2\ell+1})$ in a bigger Hilbert space and proves certain approximations for the operators $Z_{j,q}$. But the approximation there is not strong enough to enable the computation of dimension spectrum. Here we prove stronger versions of those approximations, which have made it possible to use them to compute the dimension spectrum in the previous subsection.

We start with a few simple lemmas that we will use repeatedly during the computations in this subsection.

Lemma 4.26 *Let $A \subseteq B \subseteq \Sigma$. Then one has $OP_{D_B}^{-\infty} \otimes \mathcal{E}_{\Sigma \setminus B}^{\infty} \subseteq OP_{D_A}^{-\infty} \otimes \mathcal{E}_{\Sigma \setminus A}^{\infty}$.*

Proof: Since

$$OP_{D_B}^{-\infty} = \mathcal{S}(\mathcal{H}_B) = \mathcal{S}(\mathcal{H}_A) \otimes \mathcal{S}(\mathcal{H}_{B \setminus A}) = OP_{D_A}^{-\infty} \otimes \mathcal{S}(\mathcal{H}_{B \setminus A}),$$

and $\mathcal{S}(\mathcal{H}_{B \setminus A}) \subseteq \mathcal{E}_{B \setminus A}^{\infty}$, we have the required inclusion. □

Let $A \subseteq \Sigma$. Let $\mathcal{P}$ be a polynomial in $|A|$ variables and let T be the operator on $\mathcal{H}_A$ given by

$$Te_\gamma = \mathcal{P}(\{\gamma_i, i \in A\}) q^{|\gamma_A|} e_\gamma.$$

Since the function $\gamma \mapsto \mathcal{P}(\{\gamma_i, i \in A\}) q^{|\gamma_A|}$ is a rapid decay function on Γ_A, it follows that $T \in OP_{D_A}^{-\infty}$.

Lemma 4.27 *Let $A \subseteq \Sigma$. Let T and T_0 be the following operators on $\mathcal{H}_A$:*

$$Te_\gamma = q^{\phi(\gamma_A)} Q(\psi(\gamma_A)) e_\gamma, \qquad T_0 e_\gamma = q^{\phi(\gamma_A)} e_\gamma,$$

where ϕ and ψ are some nonnegative functions. If $\phi(\gamma_A) + \psi(\gamma_A) > |\gamma_A|$, then Then $T - T_0 \in OP_{D_A}^{-\infty}$.

Proof: This is a consequence of the inequality $|1 - (1-x)^{\frac{1}{2}}| < x$ for $0 \le x \le 1$. □

Lemma 4.28 *Let $A \subseteq \Sigma$. Let T and T_0 be operators on $\mathcal{H}_A$ given by:*

$$Te_\gamma = q^{\phi(\gamma_A)} Q(\psi(\gamma_A))^{-1} e_\gamma, \qquad T_0 e_\gamma = q^{\phi(\gamma_A)} e_\gamma$$

for some nonnegative functions ϕ and ψ. If $\phi(\gamma_A) + \psi(\gamma_A) > |\gamma_A|$, then Then $T - T_0 \in OP_{D_A}^{-\infty}$.

Proof: For $0 < r < 1$, one has

$$|1 - (1-x)^{-\frac{1}{2}}| < cx \quad \text{for } 0 \le x \le r,$$

where c is some fixed constant that depends on r. Using this, it follows that the map $\gamma \mapsto q^{\phi(\gamma)}|1 - (1 - q^{2\psi(\gamma)})^{-\frac{1}{2}}|$ is a rapid decay function on Γ_A. □

For $j \in \Sigma$, we will denote by $\mathcal{E}_j$ the C^*-algebra $\mathcal{T}$ if $j \ne \ell + 1$ and the C^*-algebra $\mathcal{E}$ if $j = \ell + 1$. Thus $\mathcal{E}_j^{\infty}$ will be $\mathcal{T}^{\infty}$ for $j \ne \ell + 1$ and $\mathcal{E}^{\infty} = \mathcal{B}$ for $j = \ell + 1$. Thus $\mathcal{E}_{\Sigma}^{\infty}$ will stand for the space $\mathcal{T}_{\ell}^{\infty} \otimes \mathcal{E}^{\infty} \otimes \mathcal{T}_{\ell}^{\infty}$. Note that for any subset A of Σ, one has $OP_{D_A}^{-\infty} \subseteq \mathcal{E}_{\Sigma}^{\infty}$.

Lemma 4.29 *Let $A \subseteq \Sigma$, $a, b, m, n \in \mathbb{N}$ and $n > 0$. Let T_1 and T_2 be the operators on $\mathcal{H}_{\Sigma}$ given by*

$$T_1 e_\gamma = Q(|\gamma_A| + a(\gamma_{\ell+1})_+ + b(\gamma_{\ell+1})_- + m) e_\gamma, \qquad T_2 e_\gamma = Q(|\gamma_A| + a(\gamma_{\ell+1})_+ + b(\gamma_{\ell+1})_- + n)^{-1} e_\gamma.$$

Then T_1 and T_2 are in $\mathcal{E}_{\Sigma}^{\infty}$.

Proof: First note that if T_1' and T_1'' are operators given by

$$T_1'e_\gamma = Q(|\gamma_A| + a|\gamma_{\ell+1}| + m)e_\gamma, \qquad T_1''e_\gamma = Q(|\gamma_A| + b|\gamma_{\ell+1}| + m)e_\gamma,$$

then $T_1 = P_\Sigma T_1' + (I - P_\Sigma)T_1''$, where $P_\Sigma = \frac{I+F_\Sigma}{2}$. By the two previous lemmas, $I - T_1'$ and $I - T_2'$ are in $OP_{D_B}^{-\infty}$ where $B = A \cup \{\ell + 1\}$. Since $OP_{D_B}^{-\infty}$ is contained in $\mathcal{E}_\Sigma^\infty$, it follows that $T_1', T_2' \in \mathcal{E}_\Sigma^\infty$. Since $P_\Sigma \in \mathcal{E}_\Sigma^\infty$, we get $T_1 \in \mathcal{E}_\Sigma^\infty$.

Proof for T_2 is exactly similar. $\qquad\square$

We next proceed with a detailed analysis of the operators $Z_{j,q}$. First recall that

$$U^*e_\gamma = \xi_\gamma' = (-1)^{\sum_{i=1}^\ell (i-1)\gamma_i + \ell(\gamma_{\ell+1})_+} e_{\mathbf{r}^{n,k},\mathbf{s}}, \tag{4.41}$$

where $\mathbf{s}$ is given by

$$n = \sum_{i=\ell+2}^{2\ell+1} \gamma_i, \quad k = \sum_{i=1}^\ell \gamma_i + (\gamma_{\ell+1})_+, \tag{4.42}$$

$$d_m = \sum_{i=1}^{m-1} \gamma_i, \qquad c_m = \sum_{i=1}^\ell \gamma_i + |\gamma_{\ell+1}| + \sum_{i=\ell+2}^{2\ell+2-m} \gamma_i \quad \text{for } 1 \le m \le \ell. \tag{4.43}$$

$$d_{\ell+1} = c_{\ell+1} = \sum_{i=1}^\ell \gamma_i + (\gamma_{\ell+1})_-. \tag{4.44}$$

We will use this correspondence between $e_{\mathbf{r}^{n,k},\mathbf{s}}$ and ξ_γ' freely in what follows.

From equation (4.25), we get

$$\pi(u_{1j})e_{\mathbf{r}^{n,k}\mathbf{s}} = \sum_{M\in\mathbb{M}_j^+} C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}$$

$$+ \sum_{M\in\mathbb{M}_j^-} C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}.$$

Therefore

$$Z_{j,q}^* e_{\mathbf{r}^{n,k}\mathbf{s}} = q^{-j+1}\sum_{M\in\mathbb{M}_j^+} C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}$$

$$+q^{-j+1}\sum_{M\in\mathbb{M}_j^-} C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}.$$

Thus we have $Z_{j,q}^* = \sum_{M\in\mathbb{M}_j^+} S_M^+ T_M^+ + \sum_{M\in\mathbb{M}_j^-} S_M^- T_M^-$, where the operators $S_M^\pm$ and $T_M^\pm$ are given by

$$S_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}, \quad M \in \mathbb{M}_j^+, \tag{4.45}$$

$$S_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}, \quad M \in \mathbb{M}_j^-, \tag{4.46}$$

$$T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = q^{-j+1}C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n,k}\mathbf{s}}, \quad M \in \mathbb{M}_j^+, \tag{4.47}$$

$$T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = q^{-j+1}C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k}\mathbf{s}}, \quad M \in \mathbb{M}_j^-. \tag{4.48}$$

Lemma 4.30 *Let $S_M^\pm$ be as above. Then $US_M^\pm U^* \in \mathcal{E}_\Sigma^\infty$.*

Proof: Let us first look at the case $M \in \mathbb{M}_j^\pm$ where $1 \le j \le \ell$. In this case, one has $S_M^\pm \xi_\gamma = \xi_{\gamma'}$ where γ' is given by

$$
\gamma_i' = \begin{cases}
\gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \end{cases} \\[2em]
\gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_i = \ell + 2 - i \text{ and } i = j, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \end{cases} \\[2em]
\gamma_i & \text{otherwise.}
\end{cases}
$$

Note that since $1 \le j \le \ell$, we have $\gamma_{\ell+1}' = \gamma_{\ell+1}$, and $\eta(\gamma') - \eta(\gamma)$ depends just on M and not on γ. Therefore $US_M^\pm U^*$ is a constant times simple tensor product of shift operators. Thus in this case $US_M^\pm U^* \in \mathscr{T}_\ell^\infty \otimes I \otimes \mathscr{T}_\ell^\infty \subseteq \mathcal{E}_\Sigma^\infty$.

Next we look at the case $M \in \mathbb{M}_{\ell+1}^\pm$. In this case, define γ' and γ'' as follows:

$$
\gamma_i' = \begin{cases}
\gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \end{cases} \\[2em]
\gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \\ i = \ell + 1 \end{cases} \\[2em]
\gamma_i & \text{otherwise.}
\end{cases}
$$

$$\gamma_i'' = \begin{cases} \gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \\ i = \ell, \end{cases} \\[2em] \gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \\ i = \ell + 1 \end{cases} \\[2em] \gamma_i & \text{otherwise.} \end{cases}$$

Then one has

$$S_M^\pm \xi_\gamma = \begin{cases} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} \le 0, \\ \xi_{\gamma''} & \text{if } \gamma_{\ell+1} > 0. \end{cases}$$

Therefore in this case, one will have $U S_M^\pm U^* \in \mathscr{T}_\ell^\infty \otimes \mathcal{E}^\infty \otimes \mathscr{T}_\ell^\infty \subseteq \mathcal{E}_\Sigma^\infty$.

$\square$

We will next take a closer look at the operators $T_M^\pm$. For this, we will need to compute the quantitites involved in equations (4.47) and (4.48) more precisely than we have done earlier. We start with the computation of κ. From equation (4.4), we get

$$\begin{aligned} \psi(\mathbf{r}^{n,k}) &= -\frac{\ell}{2}(n + k + (\ell-1)k) + \frac{\ell(\ell+1)}{2}k \\ &= -\frac{\ell}{2}(n - k). \end{aligned}$$

Therefore

$$\psi(\mathbf{r}^{n,k}) - \psi(N_{1,1}(\mathbf{r}^{n,k})) = \psi(\mathbf{r}^{n,k}) - \psi(\mathbf{r}^{n+1,k}) = \frac{\ell}{2}, \tag{4.49}$$

$$\psi(\mathbf{r}^{n,k}) - \psi(N_{0,1}(\mathbf{r}^{n,k})) = \psi(\mathbf{r}^{n,k}) - \psi(\mathbf{r}^{n,k-1}) = \frac{\ell}{2}. \tag{4.50}$$

Let us write $\lambda = (n + k, k, \ldots, k, 0)$. We will next compute d_λ, where d_λ is given by (4.4). One has $d_\lambda = \sum_{\mathbf{s}} q^{2\psi(\mathbf{s})}$ where the sum is over all those $\mathbf{s}$ for which the top row is λ. Such an $\mathbf{s}$ is of the form (4.29) and one has

$$\psi(\mathbf{s}) = -\frac{1}{2}\ell(n + \ell k) + \frac{1}{2}(\ell-1)(\ell-2)k + \sum_{i=2}^{\ell}(c_i + d_i) + d_{\ell+1}.$$

Thus we have

$$d_\lambda = q^{-\ell(n+k) - 2(\ell-1)k} \sum_{\substack{k \le c_\ell \le c_{\ell-1} \le \ldots \le c_2 \le n+k \\ 0 \le d_2 \le d_3 \le \ldots \le d_\ell \le k \\ d_\ell \le d_{\ell+1} \le c_\ell}} q^{2\left(\sum_{i=2}^{\ell}(c_i + d_i) + d_{\ell+1}\right)}$$

Now for any x, we have

$$\sum_{\substack{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k\\0\le d_2\le d_3\le\ldots\le d_\ell\le k\\d_\ell\le d_{\ell+1}\le c_\ell}} x^{\left(\sum_{i=2}^\ell (c_i+d_i)+d_{\ell+1}\right)}$$

$$= \left(\sum_{k\le d_{\ell+1}\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k} x^{\left(\sum_{i=2}^\ell c_i+d_{\ell+1}\right)}\right)\left(\sum_{0\le d_2\le d_3\le\ldots\le d_\ell\le k} x^{\left(\sum_{i=2}^\ell d_i\right)}\right)$$

$$+ \left(\sum_{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k} x^{\left(\sum_{i=2}^\ell c_i\right)}\right)\left(\sum_{0\le d_2\le d_3\le\ldots\le d_\ell\le d_{\ell+1}<k} x^{\left(\sum_{i=2}^\ell d_i+d_{\ell+1}\right)}\right). \quad (4.51)$$

If we now use the identity

$$\sum_{k\le t_1\le t_2\le\ldots\le t_j\le n} x^{\left(\sum_{i=1}^j t_i\right)} = x^{jk}\prod_{i=1}^j\left(\frac{1-x^{n-k+i}}{1-x^i}\right),$$

we get

$$\sum_{\substack{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k\\0\le d_2\le d_3\le\ldots\le d_\ell\le k\\d_\ell\le d_{\ell+1}\le c_\ell}} x^{\left(\sum_{i=2}^\ell (c_i+d_i)+d_{\ell+1}\right)}$$

$$= x^{\ell k}\prod_{i=1}^\ell\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right) + x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^\ell\left(\frac{1-x^{k-1+i}}{1-x^i}\right)$$

$$= x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right)\frac{1}{1-x^\ell}\left(x^k(1-x^{n+\ell})+1-x^k\right)$$

$$= x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right)\left(\frac{1-x^{n+k+\ell}}{1-x^\ell}\right).$$

Thus

$$d_\lambda^{\frac{1}{2}} = q^{-\frac{\ell(n+k)}{2}}\prod_{i=1}^{\ell-1}\left(\frac{Q(n+i)}{Q(i)}\frac{Q(k+i)}{Q(i)}\right)\frac{Q(n+k+\ell)}{Q(\ell)}. \quad (4.52)$$

Write

$$\lambda' = (n+1+k,k,\ldots,k,0), \qquad \lambda'' = (n+k-1,k-1,\ldots,k-1,0).$$

Then one has

$$d_\lambda^{\frac{1}{2}}d_{\lambda'}^{-\frac{1}{2}} = q^{\ell/2}\frac{Q(n+1)}{Q(n+\ell)}\frac{Q(n+k+\ell)}{Q(n+k+\ell+1)},$$

$$d_\lambda^{\frac{1}{2}}d_{\lambda''}^{-\frac{1}{2}} = q^{-\ell/2}\frac{Q(k+\ell-1)}{Q(k)}\frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}.$$

Combining these with (4.49) and (4.50), we get

$$\kappa(\mathbf{r}^{n,k}, N_{1,1}(\mathbf{r}^{n,k})) = q^\ell \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)}, \tag{4.53}$$

$$\kappa(\mathbf{r}^{n,k}, N_{0,1}(\mathbf{r}^{n,k})) = \frac{Q(k+\ell-1)}{Q(k)} \frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}. \tag{4.54}$$

Lemma 4.31 Let $M \in \mathbb{M}_j^+$ and T_M^+ be as in equation (4.47). Then $UT_M^+U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ if $j \leq \ell$ or if $j = \ell+1$ and $M \neq N_\ell$.

Proof: From lemma 4.5 and equations (4.47) and (4.53), we get, for $M = (m_1, \ldots, m_j) \in \mathbb{M}_j^+$,

$$T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = \operatorname{sign}(M) q^{\ell-j+1+C(\mathbf{r}^{n,k}, N_{1,1})+B(N_{1,1})+C(\mathbf{s},M)+B(M)} \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)}$$

$$\times \quad L'(\mathbf{r}^{n,k}, 1, 1) \left(\prod_{a=1}^{j-1} L(\mathbf{s}, a, m_a, m_{a+1}) \right) L'(\mathbf{s}, j, m_j) \, e_{\mathbf{r}^{n,k}\mathbf{s}}. \tag{4.55}$$

Since $C(\mathbf{r}^{n,k}, N_{1,1}) = k$ and $B(N_{1,1}) = 0$, we get

$$T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = \operatorname{sign}(M) q^{\ell-j+1+B(M)+k+C(\mathbf{s},M)} \phi(\mathbf{s}, M) e_{\mathbf{r}^{n,k}\mathbf{s}},$$

with $\phi(\mathbf{s}, M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$ where $\psi(\gamma) = |\gamma_A| + c(\gamma_{\ell+1})_\pm + m$ for some subset $A \subseteq \Sigma$, $c \in \{0,1\}$ and some integer m that does not depend on $\mathbf{s}$. Therefore

$$UT_M^+ U^* e_\gamma = \operatorname{sign}(M) q^{\ell-j+1+B(M)+k+C(\mathbf{s},M)} \phi(\mathbf{s}, M) e_\gamma,$$

where k and $\mathbf{s}$ are given by equations (4.42–4.44). Since $\phi(\mathbf{s}, M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$, it follows from lemma 4.29 that the operator $e_\gamma \mapsto \phi(\mathbf{s}, M)e_\gamma$ is in $\mathcal{E}_\Sigma^\infty$. Next look at the operator $e_\gamma \mapsto q^{k+C(\mathbf{s},M)}e_\gamma$. Assume that there is some $i \leq j$ such that $m_i \neq 1$. Let $p = \min\{2 \leq i \leq j : m_i \neq 1\}$. Then $C(\mathbf{s}, M) \geq H_{p-1,1}(\mathbf{s}) \geq (\gamma_{\ell+1})_-$. Therefore

$$k + C(\mathbf{s}, M) \geq k + (\gamma_{\ell+1})_- = \sum_{i=1}^\ell \gamma_i + |\gamma_{\ell+1}|.$$

Hence $UT_M^+ U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Next assume that $j \leq \ell$ and $m_i = 1$ for all $i \leq j$. In this case, $C(\mathbf{s}, M) \geq H_{j,1}(\mathbf{s}) \geq (\gamma_{\ell+1})_-$. Therefore again we have

$$k + C(\mathbf{s}, M) \geq k + (\gamma_{\ell+1})_- = \sum_{i=1}^\ell \gamma_i + |\gamma_{\ell+1}|.$$

and hence $UT_M^+ U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Combining the two cases, we have the required result. $\square$

Lemma 4.32 Let $M \in \mathbb{M}_j^-$ and T_M^- be as in equation (4.48). Then $UT_M^- U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ if $M \neq N_{0,j}$.

Proof: From lemma 4.5 and equations (4.48) and (4.54), we get, for $M = (m_1, \ldots, m_j) \in \mathbb{M}_j^-$,

$$
T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = \text{sign}\,(M) q^{-j+1+C(\mathbf{r}^{n,k}, N_{0,1})+B(N_{0,1})+C(\mathbf{s},M)+B(M)} \frac{Q(k+\ell-1)}{Q(k)} \frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}
$$

$$
\times \quad L'(\mathbf{r}^{n,k}, 1, \ell+1) \left(\prod_{a=1}^{j-1} L(\mathbf{s}, a, m_a, m_{a+1}) \right) L'(\mathbf{s}, j, m_j) e_{\mathbf{r}^{n,k}\mathbf{s}}. \tag{4.56}
$$

Since $C(\mathbf{r}^{n,k}, N_{0,1}) = 0$ and $B(N_{0,1}) = 0$, we get

$$
T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = \text{sign}\,(M) q^{-j+1+C(\mathbf{s},M)+B(M)} \phi(\mathbf{s}, M) e_{\mathbf{r}^{n,k}\mathbf{s}},
$$

with $\phi(\mathbf{s}, M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$ where $\psi(\gamma) = |\gamma_A| + c(\gamma_{\ell+1})_\pm + m$ for some subset $A \subseteq \Sigma$, $c \in \{0, 1\}$ and some integer m that does not depend on $\mathbf{s}$. Therefore

$$
U T_M^- U^* e_\gamma = \text{sign}\,(M) q^{-j+1+C(\mathbf{s},M)+B(M)} \phi(\mathbf{s}, M) e_\gamma,
$$

where k and $\mathbf{s}$ are given by equations (4.42–4.44). As in the proof of lemma 4.31, it is now enough to prove that $C(\mathbf{s}, M) \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|$. Now assume that $m_i = 1$ for some $i \leq \ell$. Let $p = \min\{2 \leq i \leq j : m_i = 1\}$. Then $p \leq \ell$. We then have

$$
\begin{aligned}
C(\mathbf{s}, M) &\geq \sum_{i=1}^{p-2} H_{i,\ell+1-i}(\mathbf{s}) + H_{p-1,1}(\mathbf{s}) + H_{p-1,\ell+2-p}(\mathbf{s}) + V_{p-1,\ell+2-p}(\mathbf{s}) \\
&\geq \sum_{i=1}^{p-2} \gamma_i + (\gamma_{\ell+1})_- + \gamma_{p-1} + \left(\sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_+ - \sum_{i=1}^{p-1} \gamma_i \right) \\
&= \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|.
\end{aligned}
$$

So the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.33 As mentioned in the beginning of this subsection, weaker versions of the two lemmas above has been proved by D'Andrea in [9]. In our notation, he proves that the part of $Z_{j,q}$ that be ignored is of the order $q^k = q^{\sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_+}$, whereas we prove here that one can actually ignore terms of a slightly higher order, namely $q^{\sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|}$, which makes it possible to compute $Z_{j,q}$ modulo the ideal $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$.

Lemma 4.34 *Define operators X_j on $L_2(S_q^{2\ell+1})$ by*

$$
e_{\mathbf{r}^{n,k},\mathbf{s}} \mapsto \begin{cases} (-1)^{j-1} q^{d_j} Q(d_{j+1} - d_j) e_{\mathbf{r}^{n,k-1}, N_{0,j}(\mathbf{s})} & \text{if } 1 \leq j \leq \ell - 1, \\ (-1)^{\ell-1} q^{d_\ell} Q(d_{\ell+1} - d_\ell) Q(k - d_\ell) e_{\mathbf{r}^{n,k-1}, N_{0,\ell}(\mathbf{s})} & \text{if } j = \ell. \end{cases} \tag{4.57}
$$

Then one has

$$
U Z_{j,q}^* U^* - U X_j U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty.
$$

Proof: In view of the two forgoing lemmas, it is enough to show that

$$US^-_{N_{0,j}}T^-_{N_{0,j}}U^* - UX_jU^* \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell, \quad \text{for } 1 \le j \le \ell, \tag{4.58}$$

Let us first look at the case $1 \le j \le \ell - 1$. Observe that

$$\text{sign}(N_{0,j}) = (-1)^{j-1}, \quad C(\mathbf{r}^{n,k}, N_{0,1}) = 0 = B(N_{0,1}), \quad C(\mathbf{s}, N_{0,j}) = d_j, \quad B(N_{0,j}) = j - 1.$$

Therefore from (4.56), we get

$$
\begin{aligned}
UT^-_{N_{0,j}}U^*e_\gamma &= (-1)^{j-1}q^{d_j}\frac{Q(k+\ell-1)}{Q(k)}\frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}L'(\mathbf{r}^{n,k},1,\ell+1) \\
&\quad \times \left(\prod_{a=1}^{j-1} L(\mathbf{s},a,\ell+2-a,\ell+1-a)\right)L'(\mathbf{s},j,\ell+2-j)\,e_\gamma.
\end{aligned}
\tag{4.59}
$$

From (4.18), one gets

$$
\begin{aligned}
L'(\mathbf{r}^{n,k},1,\ell+1) &= \left(\prod_{i=2}^{\ell}\frac{Q(|k-0-i+\ell+1-1|)}{Q(|k-0-i+\ell+1|)}\right)\frac{Q(|k-0-1+\ell+1-1|)}{Q(|n+k-0-1+\ell+1|)} \\
&= \left(\prod_{i=2}^{\ell}\frac{Q(k+\ell-i)}{Q(k+\ell-i+1)}\right)\frac{Q(k+\ell-1)}{Q(n+k+\ell)} \\
&= \frac{Q(k)}{Q(n+k+\ell)}.
\end{aligned}
\tag{4.60}
$$

Similarly, from (4.15) one gets, for $1 \le a \le \ell - 1$,

$$
\begin{aligned}
&L(\mathbf{s},a,\ell+2-a,\ell+1-a) \\
&= \prod_{i=1}^{\ell+1-a}\frac{Q(|s_{a,i}-s_{a+1,\ell+1-a}-i+\ell+1-a|)}{Q(|s_{a,i}-s_{a,\ell+2-a}-i+\ell+2-a|)}\prod_{i=1}^{\ell-a}\frac{Q(|s_{a+1,i}-s_{a,\ell+2-a}-i+\ell+2-a-1|)}{Q(|s_{a+1,i}-s_{a+1,\ell+1-a}-i+\ell+1-a-1|)} \\
&= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)} \\
&\quad \times \prod_{i=2}^{\ell+1-a}\frac{Q(k-d_{a+1}-i+\ell+1-a)}{Q(k-d_a-i+\ell+2-a)}\prod_{i=2}^{\ell-a}\frac{Q(k-d_a-i+\ell+1-a)}{Q(k-d_{a+1}-i+\ell-a)} \\
&= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)} \\
&\quad \times \prod_{i=1}^{\ell-a}\frac{Q(k-d_{a+1}-i+\ell-a)}{Q(k-d_a-i+\ell+1-a)}\prod_{i=2}^{\ell-a}\frac{Q(k-d_a-i+\ell+1-a)}{Q(k-d_{a+1}-i+\ell-a)} \\
&= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)}\frac{Q(k-d_{a+1}+\ell-a-1)}{Q(k-d_a+\ell-a)},
\end{aligned}
\tag{4.61}
$$

and from (4.18), for $j \leq \ell - 1$,

$$
\begin{aligned}
L'(\mathbf{s}, j, \ell + 2 - j) \\
&= \prod_{i=1}^{\ell+1-j} \frac{Q(|s_{j+1,i} - s_{j,\ell+2-j} - i + \ell + 2 - j - 1|)}{Q(|s_{j,i} - s_{j,\ell+2-j} - i + \ell + 2 - j|)} \\
&= \frac{Q(c_{j+1} - d_j + \ell - j)}{Q(c_j - d_j + \ell + 1 - j)} \left(\prod_{i=2}^{\ell-j} \frac{Q(k - d_j + \ell + 1 - j - i)}{Q(k - d_j + \ell + 2 - j - i)} \right) \frac{Q(d_{j+1} - d_j)}{Q(k - d_j + 1)} \\
&= \frac{Q(c_{j+1} - d_j + \ell - j)}{Q(c_j - d_j + \ell + 1 - j)} \frac{Q(d_{j+1} - d_j)}{Q(k - d_j + \ell - j)}
\end{aligned}
$$

From the above two equations, we get

$$
\begin{aligned}
&\left(\prod_{a=1}^{j-1} L(\mathbf{s}, a, \ell + 2 - a, \ell + 1 - a) \right) L'(\mathbf{s}, j, \ell + 2 - j) \\
&= \frac{Q(d_{j+1} - d_j)}{Q(k + \ell - 1)} \left(\prod_{a=1}^{j-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \left(\prod_{a=1}^{j} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right).
\end{aligned}
$$

Now substituting all these in equation (4.59), we get

$$
\begin{aligned}
U T_{N_{0,j}}^{-} U^* e_\gamma &= (-1)^{j-1} q^{d_j} \frac{Q(d_{j+1} - d_j)}{Q(n + k + \ell - 1)} \left(\prod_{a=1}^{j-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \\
&\quad \times \left(\prod_{a=1}^{j} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right) e_\gamma.
\end{aligned} \tag{4.62}
$$

Now note that for $1 \leq a \leq j - 1$,

$$
d_j + c_a - d_{a+1} + \ell - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|, \qquad d_j + c_{a+1} - d_{a+1} + \ell - a - 1 \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|,
$$

and for $1 \leq a \leq j$,

$$
d_j + c_{a+1} - d_a + \ell - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|, \qquad d_j + c_a - d_a + \ell + 1 - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|,
$$

and $d_j + n + k + \ell - 1 \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|$. Therefore by using lemmas 4.27 and 4.28, we can write, modulo an operator in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$,

$$
U T_{N_{0,j}}^{-} U^* e_\gamma = (-1)^{j-1} q^{d_j} Q(d_{j+1} - d_j) \, e_\gamma.
$$

Using equation (4.46), we get

$$
U S_{N_{0,j}}^{-} U^* e_\gamma = (-1)^{j-1} e_{\gamma'},
$$

44

where

$$\gamma_i' = \begin{cases} \gamma_i & \text{if } i \neq j, \\ \gamma_i - 1 & \text{if } i = j. \end{cases}$$

Observe also that

$$U X_j U^* e_\gamma = q^{d_j} Q(d_{j+1} - d_j) \, e_{\gamma'},$$

where γ' is as above. Therefore we get (4.58) for $j \leq \ell - 1$.

In the case $j = \ell$, one has

$$L'(\mathbf{s}, \ell, 2) = \frac{Q(|s_{\ell+1,1} - s_{\ell,2}|)}{Q(|s_{\ell,1} - s_{\ell,2} + 1|)} = \frac{Q(d_{\ell+1} - d_\ell)}{Q(c_\ell - d_\ell + 1)}.$$

and as a result, one has

$$\left(\prod_{a=1}^{\ell-1} L(\mathbf{s}, a, \ell + 2 - a, \ell + 1 - a) \right) L'(\mathbf{s}, \ell, 2)$$

$$= \frac{Q(k - d_\ell)}{Q(k + \ell - 1)} \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \left(\prod_{a=1}^{\ell} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right).$$

As before, substituting all these in equation (4.59), one gets

$$\begin{aligned} U T_{N_{0,\ell}}^- U^* e_\gamma &= (-1)^{\ell-1} q^{d_\ell} \frac{Q(k - d_\ell)}{Q(n + k + \ell - 1)} \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \\ &\quad \times \left(\prod_{a=1}^{\ell} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right) e_\gamma. \end{aligned} \tag{4.63}$$

Application of lemmas 4.27 and 4.28, now enable us to write the following equality modulo an operator in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$:

$$U T_{N_{0,\ell}}^- U^* e_\gamma = (-1)^{\ell-1} q^{d_\ell} Q(k - d_\ell) Q(d_{\ell+1} - d_\ell) \, e_\gamma.$$

Using equation (4.46), we get

$$U S_{N_{0,\ell}}^- U^* e_\gamma = (-1)^{\ell-1} e_{\gamma'},$$

where

$$\gamma_i' = \begin{cases} \gamma_i & \text{if } i \neq \ell, \\ \gamma_i - 1 & \text{if } i = \ell. \end{cases}$$

Observe also that

$$U X_\ell U^* e_\gamma = q^{d_\ell} Q(k - d_\ell) Q(d_{\ell+1} - d_\ell) \, e_{\gamma'},$$

where γ' is as above. Therefore we get (4.58) for $j = \ell$. $\qquad\square$

Lemma 4.35 *Let X_j be as in lemma 4.34. Then for $1 \leq j \leq \ell$, one has $UX_jU^* - Y_{j,q}^* \otimes I \in OP_{D_\ell}^{-\infty} \otimes \mathcal{T}_\ell^\infty$.*

Proof: It follows from equations (4.41–4.44) that for $j \leq \ell-1$, one in fact has $UX_jU^* - Y_{j,q}^* \otimes I = 0$. For $j = \ell$, one has

$$\left(UX_jU^* - Y_{j,q}^* \otimes I\right) e_\gamma = \left(q^{\sum_{i=1}^{\ell-1} \gamma_i} Q\left(\gamma_\ell + (\gamma_{\ell+1})_-\right) Q\left(\gamma_\ell + (\gamma_{\ell+1})_+\right)\right) e_{\hat{\gamma}},$$

where $\hat{\gamma}_i = \gamma_i - 1$ if $i = \ell$ and $\hat{\gamma}_i = \gamma_i$ for all other i. Thus

$$\left|UX_jU^* - Y_{j,q}^* \otimes I\right| \in OP_{D_\ell}^{-\infty} \otimes \mathcal{T}_\ell^\infty \qquad \text{sign}\left(UX_jU^* - Y_{j,q}^* \otimes I\right) \in \mathcal{E}_\Sigma^\infty.$$

Therefore $UX_jU^* - Y_{j,q}^* \otimes I \in OP_{D_\ell}^{-\infty} \otimes \mathcal{T}_\ell^\infty$. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

From the two lemmas above (lemmas 4.34 and 4.35), it follows that for $1 \leq j \leq \ell$, one has $UZ_{j,q}^*U^* \in C^\infty(S_q^{2\ell+1})$. Thus we now need only to take care of the case $j = \ell + 1$.

Lemma 4.36 $UZ_{\ell+1,q}^*U^* \in C^\infty(S_q^{2\ell+1})$.

Proof: Using lemmas 4.31 and 4.32, it is enough to show that

$$U(S_{N_\ell}^+ T_{N_\ell}^+ + S_{N_{0,\ell+1}}^- T_{N_{0,\ell+1}}^-)U^* - UX_{\ell+1}U^* \quad \in \quad C^\infty(S_q^{2\ell+1}). \tag{4.64}$$

From (4.55), we get

$$T_{N_\ell}^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = q^k \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)} L'(\mathbf{r}^{n,k}, 1, 1) \left(\prod_{a=1}^{\ell} L(\mathbf{s}, a, 1, 1)\right) e_{\mathbf{r}^{n,k}\mathbf{s}}. \tag{4.65}$$

From (4.15), we get for $1 \leq a \leq \ell - 1$,

$$
\begin{aligned}
L(\mathbf{s}, & a, 1, 1) \\
&= \prod_{i=2}^{\ell+2-a} \frac{Q(|s_{a,i} - s_{a+1,1} - i + 1|)}{Q(|s_{a,i} - s_{a,1} - i + 1|)} \prod_{i=2}^{\ell+1-a} \frac{Q(|s_{a+1,i} - s_{a,1} - i + 1 - 1|)}{Q(|s_{a+1,i} - s_{a+1,1} - i + 1 - 1|)} \\
&= \prod_{i=2}^{\ell+1-a} \frac{Q(c_{a+1} - k + i - 1)}{Q(c_a - k + i - 1)} \prod_{i=2}^{\ell-a} \frac{Q(c_a - k + i)}{Q(c_{a+1} - k + i)} \\
&\quad \times \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)} \\
&= \prod_{i=1}^{\ell-a} \frac{Q(c_{a+1} - k + i)}{Q(c_a - k + i)} \prod_{i=2}^{\ell-a} \frac{Q(c_a - k + i)}{Q(c_{a+1} - k + i)} \\
&\quad \times \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)} \\
&= \frac{Q(c_{a+1} - k + 1)}{Q(c_a - k + 1)} \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)},
\end{aligned}
$$

46

and for $a = \ell$,

$$L(\mathbf{s}, \ell, 1, 1) = \frac{Q(|s_{\ell,2} - s_{\ell+1,1} - 2 + 1|)}{Q(|s_{\ell,2} - s_{\ell,1} - 2 + 1|)} = \frac{Q(d_{\ell+1} - d_\ell + 1)}{Q(c_\ell - d_\ell + 1)}.$$

Also from (4.18), we have

$$
\begin{aligned}
L'(\mathbf{r}^{n,k}, 1, 1) &= \left(\frac{\prod_{i=1}^{\ell} Q(|k - n - k - i + 1 - 1|)}{\prod_{i=2}^{\ell} Q(|k - n - k - i + 1|)} \right) \frac{1}{Q(|0 - n - k - \ell - 1 + 1|)} \\
&= \left(\prod_{i=2}^{\ell} \frac{Q(n+i)}{Q(n+i-1)} \right) \frac{Q(n+1)}{Q(n+k+\ell)} \\
&= \frac{Q(n+\ell)}{Q(n+k+\ell)}.
\end{aligned}
$$

Plugging these in equation (4.65) and using (4.41), we get

$$
\begin{aligned}
UT_{N_\ell}^+ U^* e_\gamma &= q^k \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)} \frac{Q(n+\ell)}{Q(n+k+\ell)} \\
&\quad \times \left(\prod_{a=1}^{\ell-1} \frac{Q(c_{a+1} - k + 1)}{Q(c_a - k + 1)} \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)} \right) \\
&\quad \times \frac{Q(d_{\ell+1} - d_\ell + 1)}{Q(c_\ell - d_\ell + 1)} \, e_\gamma.
\end{aligned}
$$

Thus as earlier, modulo an operator in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$, we have the equality

$$UT_{N_\ell}^+ U^* e_\gamma = q^k e_\gamma. \tag{4.66}$$

Next note that $B(N_0) = \ell$, $C(\mathbf{s}, N_0) = d_{\ell+1}$ and $\mathrm{sign}\,(N_0) = (-1)^\ell$ so that we get from (4.56)

$$
\begin{aligned}
T_{N_0}^- e_{\mathbf{r}^{n,k}\mathbf{s}} &= (-1)^\ell q^{d_{\ell+1}} \frac{Q(k + \ell - 1)}{Q(k)} \frac{Q(n + k + \ell)}{Q(n + k + \ell - 1)} \\
&\quad \times \ L'(\mathbf{r}^{n,k}, 1, \ell + 1) \left(\prod_{a=1}^{\ell} L(\mathbf{s}, a, \ell + 2 - a, \ell + 1 - a) \right) e_{\mathbf{r}^{n,k}\mathbf{s}}. \tag{4.67}
\end{aligned}
$$

Now using (4.41), (4.60), (4.61) and the fact that

$$L(\mathbf{s}, \ell, 2, 1) = \frac{Q(|s_{\ell,1} - s_{\ell+1,1} - 1 + \ell + 1 - \ell|)}{Q(|s_{\ell,1} - s_{\ell,2} - 1 + \ell + 2 - \ell|)} = \frac{Q(c_\ell - d_{\ell+1})}{Q(c_\ell - d_\ell + 1)},$$

we get

$$
\begin{aligned}
UT_{N_0}^- U^* e_\gamma &= (-1)^\ell q^{d_{\ell+1}} \frac{Q(k + \ell - 1)}{Q(k)} \frac{Q(n + k + \ell)}{Q(n + k + \ell - 1)} \frac{Q(k)}{Q(n + k + \ell)} \\
&\quad \times \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \frac{Q(k - d_{a+1} + \ell - a - 1)}{Q(k - d_a + \ell - a)} \right) \\
&\quad \times \frac{Q(c_\ell - d_{\ell+1})}{Q(c_\ell - d_\ell + 1)} \, e_\gamma.
\end{aligned}
$$

47

Thus modulo $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$, we have the equality

$$UT_{N_0}^- U^* e_\gamma = (-1)^\ell q^{d_{\ell+1}} e_\gamma. \tag{4.68}$$

Define operators $T^\pm$ on $L_2(S_q^{2\ell+1})$ by

$$T^+ \xi_\gamma = q^k \xi_\gamma, \qquad T^- \xi_\gamma = (-1)^\ell d_{\ell+1} \xi_\gamma.$$

By equations (4.66) and (4.68), it is enough to look at the operators $S_{N_\ell}^+ T^+ + S_{N_0}^- T^-$.

Now observe that

$$S_{N_0}^- \xi_\gamma = \begin{cases} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ \xi_{\gamma''} & \text{if } \gamma_{\ell+1} \leq 0, \end{cases} \qquad S_{N_\ell}^+ \xi_\gamma = \begin{cases} \xi_{\gamma'''} & \text{if } \gamma_{\ell+1} > 0, \\ \xi_{\gamma'} & \text{if } \gamma_{\ell+1} \leq 0, \end{cases}$$

where

$$\gamma_i' = \begin{cases} \gamma_i - 1 & \text{if } i = \ell + 1, \\ \gamma_i & \text{otherwise,} \end{cases} \qquad \gamma_i'' = \begin{cases} \gamma_i - 1 & \text{if } \ell \leq i \leq \ell + 2, \\ \gamma_i & \text{otherwise.} \end{cases}$$

and

$$\gamma_i''' = \begin{cases} \gamma_i + 1 & \text{if } i = \ell \text{ or } i = \ell + 2, \\ \gamma_i - 1 & \text{if } i = \ell + 1, \\ \gamma_i & \text{otherwise.} \end{cases}$$

Therefore

$$\left(S_{N_\ell}^+ T^+ + S_{N_0}^- T^- \right) \xi_\gamma = \begin{cases} q^k \xi_{\gamma'''} + (-1)^\ell q^{d_{\ell+1}} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ q^k \xi_{\gamma'} + (-1)^\ell q^{d_{\ell+1}} \xi_{\gamma''} & \text{if } \gamma_{\ell+1} \leq 0. \end{cases}$$

So if we now define

$$T\xi_\gamma = \begin{cases} (-1)^\ell q^{\sum_{i=1}^\ell \gamma_i} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ q^{\sum_{i=1}^\ell \gamma_i} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} \leq 0, \end{cases}$$

then one gets from the above equation that $U \left(S_{N_\ell}^+ T^+ + S_{N_0}^- T^- - T \right) U^*$ is in $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Thus it is enough to show that $UTU^* \in C^\infty(S_q^{2\ell+1})$. Now note that

$$\eta(\gamma) - \eta(\gamma') = \begin{cases} \ell & \text{if } \gamma_{\ell+1} > 0, \\ 0 & \text{if } \gamma_{\ell+1} \leq 0. \end{cases}$$

Therefore it follows that $UTU^* e_\gamma = q^{\sum_{i=1}^\ell \gamma_i} e_{\gamma'}$, i.e. $UTU^* = Y_{\ell+1,q}^* \otimes I$. Thus we get the required result. $\qquad \square$

Putting together lemmas 4.34, 4.35 and 4.36, we get proposition 4.24.

References

[1] S. Ališauskas and Yu. F. Smirnov. Multiplicity-free u$_q(n)$ coupling coefficients. *J. Phys. A*, 27(17):5925–5939, 1994.

[2] Partha Sarathi Chakraborty and Arupkumar Pal. Equivariant spectral triples on the quantum SU(2) group. *K-Theory*, 28(2):107–126, 2003.

[3] Partha Sarathi Chakraborty and Arupkumar Pal. Torus equivariant spectral triples for odd-dimensional quantum spheres coming from C^*-extensions. *Lett. Math. Phys.*, 80(1):57–68, 2007.

[4] Partha Sarathi Chakraborty and Arupkumar Pal. Characterization of $SU_q(\ell + 1)$-equivariant spectral triples for the odd dimensional quantum spheres. *J. Reine Angew. Math.*, 623:25–42, 2008.

[5] A. Connes and H. Moscovici. The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243, 1995.

[6] Alain Connes. Cyclic cohomology, quantum group symmetries and the local index formula for SU$_q(2)$. *J. Inst. Math. Jussieu*, 3(1):17–68, 2004.

[7] Alain Connes. On spectral characterisation of manifolds. arXiv:0810.2088 [math.OA], 2008.

[8] Ludwik Dąbrowski, Giovanni Landi, Andrzej Sitarz, Walter van Suijlekom, and Joseph C. Várilly. The Dirac operator on SU$_q(2)$. *Comm. Math. Phys.*, 259(3):729–759, 2005.

[9] Francesco D'Andrea. Noncomutative geometry and quantum group symmetries. *Ph. D. Thesis, SISSA*, arXiv:0811.3187 [math.OA], 2008.

[10] Jeong Hee Hong and Wojciech Szymański. Quantum spheres and projective spaces as graph algebras. *Comm. Math. Phys.*, 232(1):157–188, 2002.

[11] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[12] Larry B. Schweitzer. A short proof that $M_n(A)$ is local if A is local and Fréchet. *Internat. J. Math.*, 3(4):581–589, 1992.

[13] Larry B. Schweitzer. Spectral invariance of dense subalgebras of operator algebras. *Internat. J. Math.*, 4(2):289–317, 1993.

[14] Walter van Suijlekom, Ludwik Dąbrowski, Giovanni Landi, Andrzej Sitarz, and Joseph C. Várilly. The local index formula for SU$_q(2)$. *K-Theory*, 35(3-4):375–394 (2006), 2005.

[15] S. L. Woronowicz. Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted SU(N) groups. *Invent. Math.*, 93(1):35–76, 1988.

ARUPKUMAR PAL (`arup@isid.ac.in`)

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA

S. SUNDAR (`ssundar@imsc.res.in`)

Institute of Mathematical Sciences, CIT Campus, Chennai–600 113, INDIA