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Construction of some families of nested orthogonal arrays

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Abstract

A (symmetric) nested orthogonal array is a symmetric orthogonal array $OA(N, k, s, g)$ which contains an $OA(M, k, r, g)$ as a subarray, where $M < N$ and $r < s$. In this communication, some methods of construction of nested symmetric orthogonal arrays are given. Asymmetric nested orthogonal arrays are defined and a few methods of their construction are described.

Keywords: Symmetric and asymmetric nested arrays; Bose-Bush method.

1. Introduction

A symmetric orthogonal array $OA(N, k, s, g)$ is an $N \times k$ matrix with symbols from a finite set of $s \geq 2$ symbols, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ submatrix, $2 \leq g < k$. Orthogonal arrays have been studied extensively and for a comprehensive account, a reference may be made to Hedayat et al. [3].

A symmetric nested orthogonal array, $NOA((N, M), k, (s, r), g)$, where $M < N$ and $r < s$, is an $OA(N, k, s, g)$ which contains an $OA(M, k, r, g)$ as a subarray. Nested orthogonal arrays are useful in practice for designing an experimental setup consisting of two experiments, the expensive one of higher accuracy being nested in a larger and relatively less expensive one of lower accuracy. The higher accuracy experiment can, for instance, correspond to a physical experiment while the lower accuracy one can be a computer experiment. While some progress in the modeling and analysis of data from such nested experiments has been made (see e.g., Kennedy and O'Hagan [4], Reese et al. [8], Qian et al. [6] and Qian and Wu [7]), relatively less is known on the designing aspects. Nested orthogonal arrays provide an option for designing nested experiments.

The question of existence of symmetric nested orthogonal arrays has recently been examined thoroughly by Mukerjee et al. [5], who also provide some examples of such arrays. However, the construction of nested orthogonal arrays does not seem to have been studied systematically. The purpose of this article is to provide some methods of construction of (symmetric) nested orthogonal arrays. We also define an asymmetric nested orthogonal array and provide a few methods of their construction.

2. Construction of symmetric nested orthogonal arrays

Throughout, for a positive integer m, $\mathbf{0}_m$, $\mathbf{1}_m$, I_m and J_m will respectively, denote an $m\times 1$ null vector, an $m \times 1$ vector of all ones, an identity matrix of order m and an $m \times m$ matrix of all ones. Also, for a prime or a prime power u, $GF(u)$ will denote the Galois field of order u and a prime over a matrix (or vector) will denote its transpose.

Theorem 1. *Let $s > 2$ be a power of 2. Then the following families of symmetric NOAs exist:*
(a) $NOA((s^g, 2^g), g + 1,(s, 2), g)$, $g \geq 2$.
(b) $NOA((s^u, 2^u), 2u,(s, 2), 3)$, where $u \geq 4$ is an integer.
(c) $NOA((s^5, 2^5), 6,(s, 2), 4)$.

Furthermore, (i) $g+1$ is the maximum number of columns that the arrays in (a) above can accommodate, (ii) if $s = 4 = u$, then $2u$ is the maximum number of columns that the arrays in (b) above can accommodate and (iii) $k = 6$ is the maximum number of columns that the arrays in (c) above can accommodate.

Proof. (a) Let $s > 2$ be a power of 2 and define the $g \times (g + 1)$ matrix $A_1 = [I_g \;\; \mathbf{1}_g]$, where $g \geq 2$ is an integer. Then, it is easily seen that any $g \times g$ submatrix of A_1 has rank g over $GF(s)$. It follows then from Bose and Bush [1] that $C = B_1A_1$ is a (symmetric) orthogonal array $OA(s^g, g + 1, s, g)$, where B_1 is an $s^g \times g$ matrix having rows as all possible g-plets with entries from $GF(s)$. The result in (a) follows by noting that there is a $2^g \times g$ submatrix of C with elements 0 and 1. Finally, for a $NOA((s^g, 2^g), k,(s, 2), g)$ to exist, it is necessary that an $OA(2^g, k, 2, g)$ exists. It is well known that for an $OA(2^g, k, 2, g)$, $k \leq 3$ if $g = 2$ and $k \leq g+1$ if $g \geq 3$ (see e.g., Theorem 2.19 in Hedayat et al. [3]) and thus, $g + 1$ is the maximum number of columns that a nested array in (a) can accommodate for all $g \geq 2$ and this upper bound is attained.

(b) For an integer $u \geq 4$, define a $u \times 2u$ matrix $A_2 = [I_u \;\; J_u - I_u]$. Then, it can be verified that any $u \times 3$ submatrix of A_2 has rank 3 over $GF(s)$. Now, as in the proof of (a) above, one gets the required nested array by forming the product B_2A_2, where B_2 is a $2^u \times u$ matrix having rows as all possible u-plets with entries from $GF(s)$. The assertion about $2u$ being the maximum number of columns for $s = 4 = u$ follows from Theorem 2 of Mukerjee et al. [5].

(c) Let $A_3 = [I_5 \;\; \boldsymbol{a}]$, where $\boldsymbol{a} = (1, 0, 1, 1, 1)'$. Then, it can be verified that any 5×4 submatrix of A_3 has rank 4 over $GF(s)$. As in the proof of part (a), the required nested array is given by $C = B_3A_3$, where B_3 is an $s^5 \times 5$ matrix having rows as all possible 5-plets with elements from $GF(s)$. The assertion that $k \leq 6$ for the nested arrays in (c) follows from the fact that in an (ordinary) orthogonal array $OA(32, k, 2, 4)$, $k \leq 6$ (Seiden and Zemach [9]). □

Theorem 2. *If $s > 4$ is a power of 2, then there exists a symmetric $NOA((s^2, 4^2), 3, (s, 4), 2)$.*

Proof. The proof follows essentially on the lines of that of Theorem 1 by considering the matrix $A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, noting that (i) any 2×2 submatrix of A_4 has rank 2 over $GF(s), s = 2^t, t \geq 3$, (ii) there is a $4^2 \times 2$ submatrix of B_4 with elements $0, 1, x, x+1$ only, where B_4 is an $s^2 \times 2$ matrix with elements from $GF(s)$ and (iii) the fact that the elements of $GF(s)$ are $0, 1$ and all polynomials (in x) of degree at most $t-1$. □

Theorem 3. *If $s \geq 3$ is an integer such that both $s-1$ and $s+1$ are prime powers, then there is a symmetric $NOA((2s^2, (s-1)^2), s, (s, s-1), 2)$. Furthermore, the number of columns in such an array is bounded above by s.*

Proof. Step 1: Construct an $OA((s+1)^2, s+2, s+1, 2)$, say A, by utilizing a complete set of mutually orthogonal Latin squares of side $s+1$ and let the symbols of this array be $0, 1, \ldots, s$. Then, it is not hard to see that, upto isomorphism, this array has two rows, $(0, 0, \ldots, 0)$ and $(1, 0, 1, 1, \ldots 1)$. In A, replace every 1 by 0, delete the two rows consisting of all zeros and delete the first two columns to arrive at an $(s^2 + 2s - 1) \times s$ array, say A_1. Note that A_1 involves the s symbols $0, 2, \ldots, s$.

Step 2: Construct an $OA((s-1)^2, s, s-1, 2)$ involving $s-1$ symbols, $2, \ldots, s$ and call this array A_2.

Step 3: Consider the $2s^2 \times s$ matrix $B = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$.

Then, arguing as in Hedayat et al. [3, p.243], one can show that B is an $OA(2s^2, s, s, 2)$. Note that our construction of the orthogonal array B is slightly different from that of Hedayat et al.

The claim in Theorem 3 is now immediate by noting that the orthogonal array A_2 is precisely the smaller array in the nested array. Finally, since in an $OA((s-1)^2, k, (s-1), 2)$, $k \leq s$, the assertion about the upper bound on the number of columns in the constructed nested array follows. □

3. Asymmetric Nested Orthogonal Arrays

So far, we have restricted attention to symmetric nested orthogonal arrays. We now introduce asymmetric nested orthogonal arrays.

Definition. *An asymmetric nested orthogonal array, $NOA((N, M), k, (s_1 \times s_2 \times \cdots \times s_k, r_1 \times r_2 \times \cdots \times r_k), g)$, where $r_i \leq s_i$, with strict inequality for at least one i, $1 \leq i \leq k$, and*

$M < N$, *is an asymmetric orthogonal array,* $OA(N, k, s_1 \times \cdots \times s_k, g)$ *which contains an* $OA(M, k, r_1 \times \cdots \times r_k, g)$ *as a subarray.*

Remark. Note that the above definition does not preclude the possibility of existence of an asymmetric nested orthogonal array wherein the smaller orthogonal array is a symmetric orthogonal array, nested within a larger asymmetric orthogonal array. For example, consider the following array, displayed in transposed form:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}'.$$

The first 8 rows of this array form a (symmetric) orthogonal array $OA(8, 4, 2, 3)$ while all the 16 rows represent an asymmetric orthogonal array $OA(16, 4, 4 \times 2 \times 2 \times 2, 3)$. We continue to call such arrays also as asymmetric nested orthogonal arrays.

We now describe some methods of construction of asymmetric nested orthogonal arrays.

Theorem 4. *The existence of an* $OA(N, k, 2, 2u)$*, where* $u \geq 1$ *is an integer implies the existence of an* $NOA((tN, 2mN), k+1, (t^1 \times 2^k, (2m)^1 \times 2^k), 2u+1)$*, where* $t \geq 2$ *is an even integer and* m $(1 \leq m < t)$ *is an integer.*

Proof. Let A denote an $OA(N, k, 2, 2u)$, with symbols 0 and 1 (without loss of generality) and let $\bar{A}$ denote the $N \times k$ matrix obtained by interchanging the two symbols in A. Consider the $tN \times (k+1)$ array B, given by

$$B = \begin{bmatrix} \mathbf{0}'_N & \mathbf{1}'_N & 2\mathbf{1}'_N & 3\mathbf{1}'_N & \dots & (t-2)\mathbf{1}'_N & (t-1)\mathbf{1}'_N \\ \\ A' & \bar{A}' & A' & \bar{A}' & \cdots & A' & \bar{A}' \end{bmatrix}'.$$

Then, it can easily be verified that B is an $OA(tN, k+1, t^1 \times 2^k, 2u+1)$. The array

$$\begin{bmatrix} \mathbf{0}'_N & \mathbf{1}'_N & 2\mathbf{1}'_N & 3\mathbf{1}'_N & \dots & (m-2)\mathbf{1}'_N & (m-1)\mathbf{1}'_N \\ \\ A' & \bar{A}' & A' & \bar{A}' & \cdots & A' & \bar{A}' \end{bmatrix}',$$

nested within B, is an $OA(2mN, k+1, (2m)^1 \times 2^k, 2u+1)$, where m, $1 \leq m < t$, is an integer. □

Example 1. Considering A to be an $OA(4, 3, 2, 2)$, taking $t = 6, m = 2$ and following the above method of construction, one obtains an $NOA((24, 16), 4, (6 \times 2^3, 4 \times 2^3), 3)$ which is displayed below in transposed form:

$$\begin{bmatrix} 0000 & 1111 & 2222 & 3333 & 4444 & 5555 \\ 0011 & 1100 & 0011 & 1100 & 0011 & 1100 \\ 0101 & 1010 & 0101 & 1010 & 0101 & 1010 \\ 0110 & 1001 & 0110 & 1001 & 0110 & 1001 \end{bmatrix}'.$$

The first 16 rows of the above array form an asymmetric $OA(16, 4, 4 \times 2^3, 3)$ and the full array is an $OA(24, 4, 6 \times 2^3, 3)$.

Next, consider an asymmetric orthogonal array $A = OA(N, k, s_1 \times s_2 \times \cdots \times s_k, g)$, where $g \geq 2$ and suppose t is a positive integer such that $s_1 | t$. Write A as

$$A = \begin{bmatrix} \boldsymbol{a}_1' & \boldsymbol{a}_2' & \cdots & \boldsymbol{a}_{s_1}' \\ A_1' & A_2' & \cdots & A_{s_1}' \end{bmatrix}',$$

where for $1 \leq i \leq s_1$, $\boldsymbol{a}_i$ is an $N/s_1 \times 1$ vector with each element equal to i. Clearly, each A_i $(1 \leq i \leq s_1)$ is an $OA(N/s_1, k-1, s_2 \times \cdots \times s_k, g-1)$. Define $u = N/s_1$, $v = t/s_1, \boldsymbol{b} = (0, 1, \ldots, t-1)'$ and $A^* = [A_1' \vdots A_2' \vdots \cdots \vdots A_{s_1}']'$. Consider the matrix B given by

$$B = [\boldsymbol{b} \otimes \mathbf{1}_u \vdots \mathbf{1}_v \otimes A^*],$$

where $\otimes$ stands for the Kronecker (tensor) product of matrices. Then, one can easily see that B is an asymmetric nested array $NOA((Nt/s_1, N), k, (t \times s_2 \times \cdots \times s_k, s_1 \times s_2 \times \cdots \times s_k), g)$, where the first N rows of B form the smaller array, which is an $OA(N, k, s_1 \times s_2 \times \cdots \times s_k, g)$. We thus have the following result.

Theorem 5. *Suppose an orthogonal array $OA(N, k, s_1 \times s_2 \times \cdots \times s_k, g)$, where $g \geq 2$, is available and suppose t is a positive integer such that $s_1 | t$. Then there exists an $NOA((Nt/s_1, N), k, (t \times s_2 \times \cdots \times s_k, s_1 \times s_2 \times \cdots \times s_k), g)$.*

Example 2. Consider an $OA(16, 9, 4^3 \times 2^6, 2)$, A, displayed in transposed form below:

$$A = \begin{bmatrix} 0000 & 1111 & 2222 & 3333 \\ 0123 & 0123 & 0123 & 0123 \\ 2301 & 3210 & 0123 & 1032 \\ 0011 & 1100 & 1100 & 0011 \\ 1001 & 0110 & 1001 & 0110 \\ 0101 & 0101 & 1010 & 1010 \\ 1010 & 0101 & 0101 & 1010 \\ 1001 & 1001 & 0110 & 0110 \\ 1100 & 0011 & 1100 & 0011 \end{bmatrix}'.$$

Using the above (ordinary) orthogonal array, choosing $s_1 = 4, t = 8$ and following the construction described above, we have an $NOA((32, 16), 9, (8 \times 4^2 \times 2^6, 4^3 \times 2^6), 2)$, shown below in transposed form:

$$\begin{bmatrix} 0000 & 1111 & 2222 & 3333 & 4444 & 5555 & 6666 & 7777 \\ 0123 & 0123 & 0123 & 0123 & 0123 & 0123 & 0123 & 0123 \\ 2301 & 3210 & 0123 & 1032 & 2301 & 3210 & 0123 & 1032 \\ 0011 & 1100 & 1100 & 0011 & 0011 & 1100 & 1100 & 0011 \\ 1001 & 0110 & 1001 & 0110 & 1001 & 0110 & 1001 & 0110 \\ 0101 & 0101 & 1010 & 1010 & 0101 & 0101 & 1010 & 1010 \\ 1010 & 0101 & 0101 & 1010 & 1010 & 0101 & 0101 & 1010 \\ 1001 & 1001 & 0110 & 0110 & 1001 & 1001 & 0110 & 0110 \\ 1100 & 0011 & 1100 & 0011 & 1100 & 0011 & 1100 & 0011 \end{bmatrix}'.$$

The first 16 rows of the above array form an $OA(16, 9, 4^3 \times 2^6, 2)$ while the full array is an $OA(32, 9, 8 \times 4^2 \times 2^6, 2)$.

Numerous applications of Theorem 5 can be made to obtain asymmetric nested orthogonal arrays. For example, let $N \geq 4$ and $T \leq N$ be Hadamard numbers, where a positive integer $u \geq 2$ is called a Hadamard number if a Hadamard matrix of order u exists. Then there exists an $OA(NT, NT - 2T + 1, 4^{T-1} \times 2^{NT-3T+2}, 2)$ exists (Cheng [2]). Using this orthogonal array, one gets an $NOA((NTS/4, NT), NT - 2T + 1, (S \times 4^{T-2} \times 2^{NT-3T+2}), 2)$, where S is a multiple of 4. Such examples can be multiplied. Details are omitted.

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