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Extensions of Schur's irreducibility results

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EXTENSIONS OF SCHUR'S IRREDUCIBILITY RESULTS

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Abstract. We prove that the generalised Laguerre polynomials $L_n^{(\alpha)}(x)$ with $0 \le \alpha \le 50$ are irreducible except for finitely many pairs (n, α) and that these exceptions are necessary. In fact it follows from a more general statement.

1. Introduction

For $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$ with $n \ge 1$, we define the generalised Laguerre polynomials of degree n as

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{(n-j)!j!}.$$

There is an extensive literature on Laguerre polynomials. In particular, the irreducibility of these class of orthogonal polynomials has been well studied. The irreducibility of $L_n^{(-2n-1)}$ proved by Filaseta and Trifonov [FiTr02] is equivalent to the fact that all Bessel polynomials are irreducible. Also Laguerre polynomials provide examples of polynomials of degree n with associated Galois group A_n where A_n is the alternating group on n symbols and the irreducibility of $L_n^{(n)}$ proved by Filaseta, Kidd and Trifonov [FiKiTr] has been used to settle explicitly the *Inverse Galois problem* that for every $n > 1$ there exists an explicit polynomial of degree n with associated Galois group A_n. We prove

Theorem 1. *Let $0 \le \alpha \le 50$. Then $L_n^{(\alpha)}(x)$ is irreducible except when $n = 2, \alpha \in \{2, 7, 14, 23, 34, 47\}$ and $n = 4, \alpha \in \{5, 23\}$ where it has a linear factor.*

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For the exceptions, we have

$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6);$ $L_2^{(7)}(x) = \frac{1}{2}(x-6)(x-12);$
$L_2^{(14)}(x) = \frac{1}{2}(x-12)(x-20);$ $L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30);$
$L_2^{(34)}(x) = \frac{1}{2}(x-30)(x-42);$ $L_2^{(47)}(x) = \frac{1}{2}(x-42)(x-56);$
$L_4^{(5)}(x) = \frac{1}{24}(x-6)(x^3-30x^2+252x-504);$
$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040).$

Theorem 1 is an extension of a result of Filaseta, Finch and Leidy [FiFiLe] where they proved that $L_n^{(\alpha)}(x)$ is irreducible for all n and $0 \leq \alpha \leq 10$ except when $(n, \alpha) \in \{(2,2),(4,5),(2,7)\}$. Therefore we shall always assume that $\alpha > 10$ in the proof of Theorem 1. We also consider the problem of finding factors of Laguerre polynomials. We have

Theorem 2. *Let $1 \leq k \leq \frac{n}{2}$ and $0 \leq \alpha \leq 5k$. Then $L_n^{(\alpha)}(x)$ has no factor of degree k except when $k = 1, (n, \alpha) \in \{(2,2),(4,5)\}$.*

The Laguerre polynomials are a special case of generalizations of following class of polynomials first considered by Schur. Let $n \geq 1, a \geq 0$ and $a_0, a_1, \ldots, a_n$ be integers. The *generalized Schur polynomials* are defined as

(1)
$$f(x) := f_{n,a}(x) := f_{n,a}(a_0, a_1, \cdots, a_n) = a_n \frac{x^n}{(n+a)!} + a_{n-1}\frac{x^n}{(n-1+a)!} + \ldots + a_1 \frac{x}{(1+a)!} + a_0 \frac{1}{a!}.$$

It is easy to see that by taking

$$a = \alpha \text{ and } a_j = (-1)^j \binom{n}{j} \text{ for } 0 \leq j \leq n,$$

we obtain $(n+\alpha)! f_\alpha(x) = n! L_n^{(\alpha)}(x)$.

Schur [Sch29] proved that $f(x)$ with $a = 0$ and $|a_0| = |a_n| = 1$ is irreducible. He also proved in [Sch73] that $f(x)$ with $a = 1$ and $|a_0| = |a_n| = 1$ is irreducible unless $n + 1 = 2^r$ for some r where it may have a linear factor or $n = 8$ where it may have a quadratic factor. Also for $a = 2$ and many other values of a the polynomial $f(x)$ may have a linear factor. Clearly if $f(x)$ is reducible, then $f(x)$ has a factor of degree k with $1 \leq k \leq \frac{n}{2}$. Shorey and Tijdeman [ShTi] proved that $f(x)$ with $2 \leq k \leq \frac{n}{2}$, $0 \leq a \leq \frac{3}{2}k$ and $|a_0| = |a_n| = 1$ has no factor of degree k except when

$$(n,k,a) \in \{(6,2,3),(7,2,2),(7,2,3),(7,3,3),(8,2,1),(8,3,2), (12,3,4),(13,2,3),(22,2,3),(46,3,4),(78,2,3)\}. \tag{2}$$

Furthermore all the exceptions in (2) are necessary. They also showed that for $f(x)$ with $3 \leq k \leq \frac{n}{2}, |a_0| = |a_n| = 1$ and $0 \leq a \leq 10$ when $k = 3,4$ or $0 \leq a \leq 30$ when $k \geq 5$ has no factor of degree k except when

$$(n,k,a) \in \{(7,3,3),(8,3,2),(12,3,4),(18,4,9),(18,4,10),(46,3,4),\\ (56,4,10),(17,5,11),(19,5,9),(40,5,12)\}. \tag{3}$$

We extend the validity of their results as follows.

Theorem 3. *Let $2 \le k \le \frac{n}{2}$, $0 \le a \le 5k$ and $|a_0| = |a_n| = 1$. Then $f_{n,a}(x)$ has no factor of degree k except possibly when (n,k,a) is given by (2) or (3) or*

$$\begin{aligned} k = 2,\ (n,a) \in \{&(4,5),(6,4),(8,8),(12,4),(17,8),(21,4),(22,6),(23,5),\\ &(23,10),(24,9),(36,9),(43,6),(44,5),(46,9),(58,6),(59,5),\\ &(72,9),(73,8),(77,4),(91,9),(112,9),(233,10),(234,9)\};\\ k = 3,\ (n,a) \in \{&(14,12),(17,11),(53,12)\};\\ k = 4,\ (n,a) \in \{&(16,12),(17,11),(38,13),(39,18)\}. \end{aligned} \tag{4}$$

Theorem 4. *Let $2 \le k \le \frac{n}{2}$, $|a_0| = |a_n| = 1$ and $0 \le a \le 40$ if $k = 2$ and $0 \le a \le 50$ if $k \ge 3$. Then $f_{n,a}(x)$ has no factor of degree k except possibly when (n,k,a) is given by (2) or (3) or (4) or the cases $k = 2$ with*

$$n + a \le 100 \text{ or } a \in \{13,14,19,33\}, n + a \in \{126,225,2401,4375\}$$

or

a	$n+a$	a	$n+a$	a	$n+a$
12	169, 729	15, 16	289	17	513
18	361, 513, 1216	19, 20	243	21	529
21, 22	121, 576	24	325, 625, 676	27	784
28	145	29	961	31	243
32	243, 289, 1089	33	136, 256, 289, 5832	36	1369
38	325, 625, 676	39	1025, 6561	40	288

It is likely to obtain factorizations in most of these cases but we have not carried out the computations. The following assertion follows from Theorem 4.

Theorem 5. *The polynomial $f_{n,a}(x)$ with $a_0a_n = \pm 1, a_1 = a_2 = \ldots = a_{n-1} = 1$ and $a \le 12$ is either irreducible or a product of a linear polynomial times a polynomial of degree $n-1$. factor.*

We shall use the results of [ShTi] stated above without reference in this paper. Thus we always suppose that $a > 3$ if $k = 2$, $a > 10$ if $k = 3, 4$ and $a > 30$ if $k \ge 5$ in Theorems 3 and 4. Further we observe that Theorem 4 with $k \ge 10$ follows from Theorem 3. Also Theorem 2 follows immediately from Theorem 1 for $k \le 10$ and from Theorem 3 for $k > 10$. Thus it suffices to prove Theorems 1, 3, 4 with $k < 10$ and 5. The new ingredients in the proofs of our theorems are the following Irreducibility Lemma and sharper lower estimates for the greatest prime factor of $\Delta(m, k)$ where

$$\Delta(m,k) = m(m+1)\cdots(m+k-1). \tag{5}$$

Lemma 1.1. *Let $a > 0, 1 \leq k \leq \frac{n}{2}$ and $u_0 = \frac{a}{k}$.*

(A) Assume that there is a prime $p \geq k+2$ with

$$p | \prod_{i=1}^{k} (a+n-k+i), \quad p \nmid a_0 a_n \tag{6}$$

and

$$p \nmid \prod_{i=1}^{k} (a+i). \tag{7}$$

Suppose

$$p \geq \min(2u_0, k+u_0) \tag{8}$$

or

$$p > 2k \text{ and } p^2 - p \geq a. \tag{9}$$

Then $f_{n,a}(x)$ has no factor of degree k.

(B) If there is a prime $p \geq k+2$ with

$$p | \prod_{i=1}^{k} (n-k+i)(a+n-k+i) \tag{10}$$

and (7) and satisfying (8) or (9), then $L_n^{(a)}(x)$ has no factor of degree k.

We have stated Lemma 1.1 and some of the subsequent lemmas in a more general way than required for the proof of our theorems. We prove Lemma 1.1 in Section 2. In Section 3, we give a refinement of an argument of Erdős and Sylvester. In Sections 5−9, we prove Theorems 1, 3, 4 and 5 by combining Lemma 1.1 with the refinement in Section 4, results on Grimm's conjecture (see Lemma 3.4) and estimates from prime number theory. Section 3 contains preliminaries required for the proof of our theorems. For any real $u > 0$, let $\lfloor u \rfloor$ and $\lceil u \rceil$ be the floor function of u and the ceiling function of u, respectively. Thus $\lfloor u \rfloor$ is the greatest integer less than or equal to u and $\lceil u \rceil$ is the least integer exceeding u.

2. Proof of Lemma 1.1

We will use the notations introduced in this section throughout the paper. We write

$$\Delta_j = \Delta(a+1, j) = (a+1)(a+2)\cdots(a+j).$$

We observe that $q|\Delta_k$ for all primes $k < q \leq \frac{a+k}{\lceil u_0 \rceil}$ since $a \leq k\lceil u_0 \rceil < q\lceil u_0 \rceil \leq a+k$. Suppose there is a prime p satisfying the condition of the lemma. Then $p > \frac{a+k}{\lceil u_0 \rceil}$ by (7). As in the proof of [ShTi, Lemma 4.2], it suffices to show that

$$\phi_j := \phi_j(p) := \frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k} \text{ for } 1 \leq j \leq n \tag{11}$$

for showing that $f_{n,a}(x)$ has no factor of degree k. Also as in the proof of [FiFiLe, Lemma 2.4], for showing $L_n^{(a)}(x)$ has no factor of degree k, it suffices to show

$$\phi'_j := \phi'_j(p) := \frac{\text{ord}_p\left(\frac{\Delta_j}{\binom{n}{j}}\right)}{j} < \frac{1}{k} \text{ for } 1 \leq j \leq n. \tag{12}$$

Since $\phi'_j \leq \phi_j$, we show that (11) holds for all j.

Let j_0 be the minimum j such that $p|(a+j)$ and write $a + j_0 = pl_0$ for some l_0. Then $j_0 \leq p$ and $j_0 > k$ since $p \nmid \Delta_k$. Also we see that $l_0 \leq \lceil u_0 \rceil$ which we shall use in the proof without reference.

We may restrict to those j such that $a + j = pl$ for some l. Then $j - j_0 = p(l - l_0)$. Writing $l = l_0 + s$, we get $j = j_0 + ps$. Note that if $p|(a+j)$, then $a + j = p(l_0 + r)$ for some r. Hence we have

$$\text{ord}_p(\Delta_j) = \text{ord}_p((pl_0)(p(l_0+1))\cdots(p(l_0+s))) = s + 1 + \text{ord}_p(l_0(l_0+1)\cdots(l_0+s)). \tag{13}$$

Let r_0 be such that $\text{ord}_p(l_0 + r_0)$ is maximal. We consider two cases.

Case I: Assume that $l_0+s < p^2$. If $p \nmid (l_0+i)$ for $0 \leq i \leq s$, then $\phi_j = \frac{s+1}{j_0+ps} < \frac{s+1}{k+ks} = \frac{1}{k}$. Hence we may suppose that $p|(l_0+i)$ for some $0 \leq i \leq s$ and further $l_0 + s = pl_1$ for some $1 \leq l_1 < p$. Assume $s = 0$. Then $p|l_0$ which together with $l_0 < p^2$ implies $\text{ord}_p(\Delta_j) = \text{ord}_p(a+j_0) = 2$. Therefore $a + p \geq a + j_0 \geq p^2$ implying $a \geq p^2 - p$. If (8) holds, then $a \leq \max(k(p-k), \frac{pk}{2}) < p(p-1)$ which is not possible. Thus (9) holds and hence $p \geq 2k+1$ and $a = p^2 - p$ implying $j_0 = p$. Therefore $\phi_j = \frac{2}{j_0} = \frac{2}{p} < \frac{1}{k}$. Thus we have $s \neq 0$ and we obtain from (13) that $\text{ord}_p(\Delta_j) = s+1+l_1$ implying $\phi_j \leq \frac{s+1+l_1}{j_0+ps}$. Hence $\phi_j < \frac{1}{k}$ if $(p-k)\frac{s}{l_1} \geq k$ since $\frac{j_0+sp}{k} > 1 + s\frac{p}{k}$.

Suppose p satisfies (9). Then we may assume that $s < l_1$. Since $l_1 < p$, we have $s < p$ implying $\text{ord}_p(\Delta_j) \leq s + 2$ giving $\phi_j < \frac{s+2}{k+ps} \leq \frac{1}{k}$ since $s > 0$.

Thus we assume that p satisfies (8). Since $p \geq k+2, s = pl_1 - l_0$ and $l_0 \leq \lceil u_0 \rceil$, we have $(p-k)\frac{s}{l_1} - k \geq 2(p - \frac{l_0}{l_1}) - k \geq 2p - k - 2\lceil u_0 \rceil$. Hence it suffices to show $2p - k \geq 2\lceil u_0 \rceil$. Since $p \geq \min(2u_0, k+u_0)$, we have

$$2p - k = p + p - k \geq \begin{cases} 2u_0 + 2 \geq 2\lceil u_0 \rceil & \text{if } p \geq 2u_0 \\ 2(k + \lceil u_0 \rceil) - k \geq 2\lceil u_0 \rceil & \text{if } p \geq k + u_0, \end{cases}$$

noting that $p \geq k + u_0$ implies $p \geq k + \lceil u_0 \rceil$.

Case II: Let $l_0 + s \geq p^2$. Then we get from (13) that

$$\mathrm{ord}_p(\Delta_j) \leq s + 1 + \mathrm{ord}_p(l_0 + r_0) + \mathrm{ord}_p(s!) \leq s + 1 + \frac{\log(l_0 + s)}{\log p} + \frac{s}{p-1}.$$

Since $\frac{j}{k} = \frac{j_0 + ps}{k} > 1 + \frac{p}{k}s$, it is enough to show that

$$\frac{p}{k} \geq 1 + \frac{1}{p-1} + \frac{\log(l_0 + s)}{s \log p}.$$

Observe that $\frac{\log(l_0+s)}{s \log p}$ is a decreasing function of s. Since $s \geq p^2 - l_0$, it suffices to show

$$\frac{p}{k} \geq 1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0}.$$

Suppose p satisfies (8). Then from $l_0 \leq \lceil u_0 \rceil \leq p$ and $p \geq k+2$, we have $p^2 - l_0 \geq (k+2)^2 - (k+2) \geq 2(k+1)$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leq 1 + \frac{1}{k+1} + \frac{2}{2(k+1)} < 1 + \frac{2}{k} \leq \frac{p}{k}.$$

Suppose p satisfies (9). Then from $l_0 \leq \lceil u_0 \rceil \leq a$ and $p > 2k$, we obtain $p^2 - l_0 \geq p^2 - a \geq p > 2k$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leq 1 + \frac{1}{2k} + \frac{2}{2k} < 1 + \frac{2}{k} \leq \frac{p}{k}.$$

Hence the assertion. □

Corollary 2.1. *Let k, p and $\mathfrak{A}_{k,p}$ be given by*

$$k=1,\ p=3,\ \mathfrak{A}_{1,3}=\{3r,3r+1:0\le r\le 16\}\setminus\{7,16,24,25,34,43\}$$
$$k=1,\ p=5,\ \mathfrak{A}_{1,5}=\{5r,5r+1,5r+2,5r+3:0\le r\le 9\}\cup\{50\}\setminus\{23,48\}$$
$$k=1,\ p=7,\ \mathfrak{A}_{1,7}=[0,50]\cap\mathbb{Z}\setminus\{6,13,20,27,34,41,47,48\}$$
$$k=2,\ p=5,\ \mathfrak{A}_{2,5}=\{5r,5r+1,5r+2:0\le r\le 8\}\cup\{45,50\}\setminus\{21,22\}$$
$$k=2,\ p=7,\ \mathfrak{A}_{2,7}=[0,50]\cap\mathbb{Z}\setminus(\{7r-1,7r-2:1\le r\le 7\}\cup\{45,46\})$$
$$k=3,\ p=5,\ \mathfrak{A}_{3,5}=\{0,1,5,6,10,11,15,25,26,30,31,35,36,40,50\}$$
$$k=3,\ p=7,\ \mathfrak{A}_{3,7}=\{7r,7r+1,7r+2,7r+3:0\le r\le 5\}\cup\{42,49,50\}$$
$$k=4,\ p=7,\ \mathfrak{A}_{4,7}=\{7r,7r+1,7r+2:0\le r\le 4\}\cup\{35,36,49,50\}$$
$$k=5,\ p=7,\ \mathfrak{A}_{5,7}=\{0,1,7,8,14,15,21,22,28,49,50\}.$$

Suppose $n\ge 2k$ and p satisfies (6). *Then $f_{n,a}(x)$ has no factor of degree k for $a\in\mathfrak{A}_{k,p}$. Further if p satisfies* (10), *then $L_n^{(a)}(x)$ has no factor of degree k for $a\in\mathfrak{A}_{k,p}$.*

Proof. For k,p and $a\in\mathfrak{A}_{k,p}$ given in the statement of Corollary 2.1, we check that $p\nmid\Delta_k$ and $\frac{\mathrm{ord}_p(\Delta_j)}{j}<\frac{1}{k}$ for $j\le 50$. As in the proof of Lemma 1.1, it suffices to check that $\frac{\mathrm{ord}_p(\Delta_j)}{j}<\frac{1}{k}$ for all $j\ge 1$. Since $\mathrm{ord}_p(s!)\le\frac{s}{p-1}$, we have for $j>50$ that

$$\frac{\mathrm{ord}_p(\Delta_j)}{j}=\frac{\mathrm{ord}_p((a+j)!)-\mathrm{ord}_p(a!)}{j}\le\frac{\frac{a+j}{p-1}-\mathrm{ord}_p(a!)}{j}\le\frac{1}{p-1}+\frac{\frac{a}{p-1}-\mathrm{ord}_p(a!)}{51}<\frac{1}{k}.$$

Thus $\frac{\mathrm{ord}_p(\Delta_j)}{j}<\frac{1}{k}$ for all $j\ge 1$. □

Corollary 2.2. *Let $a>0$ and $1\le k\le\frac{n}{2}$.*

(*i*) *If there is a prime $p>a+k$ satisfying* (6), *then $f_{n,a}(x)$ has no factor of degree k.*

(*ii*) *Let $p\ge k+2$ be a prime satisfying* (6) *and let*

$$\mathcal{A}_p=\bigcup_{i=1}^{r_p}([ip-k,ip-1]\cap\mathbb{Z}_{>0})\cup\{j>pr_p,j\in\mathbb{Z}\}$$

where

$$r_p=\lfloor\frac{k}{2}\rfloor\ \text{ if }\ p<2k \text{ and } p-1\ \text{ if }\ p\ge 2k.$$

Then $f_{n,a}(x)$ has no factor of degree k for $a\notin\mathcal{A}_p$.

(*iii*) *Let $P_1>P_2>\ldots>P_s\ge k+2$ be primes satisfying* (6). *For a subset $\{Q_1,Q_2,\ldots,Q_g\}\subseteq\{P_1,P_2,\ldots,P_s\}$, let*

$$\mathcal{B}\{Q_1,\ldots,Q_g\}=\bigcap_{l=1}^{g}\mathcal{A}_{Q_l}.$$

Then $f_{n,a}(x)$ *has no factor of degree* k *for* $a \notin \mathcal{B}\{Q_1, \ldots, Q_g\}$.

In earlier results, Corollary 2.2 (i) has been used. This is possible only if there is a $p > k+a$ satisfying (6). But it is possible to apply Lemma 1.1 even when $p \leq k+a$ for all p satisfying (6). For example, take $n = 15, a = 13, k = 3$. Here $p < k+a$ for all p satisfying (6). However (6), (7) and (9) are satisfied with $p = 13$ and hence $f_{n,13}(x)$ has no factor of degree 3 by Lemma 1.1.

Proof. (i) is immediate from Lemma 1.1. Consider (ii). We may assume that $p \leq k+a$ by (i). Let $a \notin \mathcal{A}_p$. Then $a \leq pr_p$ implying $a \leq p^2 - p$ if $p \geq 2k$ and $2u_0 = \frac{2a}{k} \leq \frac{2pr_p}{k} \leq p$ if $p < 2k$ satisfying either (8) or (9). Since $a \notin \mathcal{A}_p$, there is some i for which $ip - 1 < a < (i+1)p - k$ implying $ip < a+1 < a+k < (i+1)p$. Therefore $p \nmid \prod_{j=1}^{k}(a+j)$ which together with (6) and $p \geq k+2$ satisfy the conditions of Lemma 1.1. Now the assertion follows by Lemma 1.1. The assertion (iii) follows from (ii). □

3. Preliminaries for Theorems 3-5

For a positive integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime factors and the greatest prime factor of ν, respectively, and we put $\omega(1) = 0, P(1) = 1$. For positive integers ν, we write

$$\pi(\nu) = \sum_{p \leq \nu} 1,$$
$$\theta(\nu) = \sum_{p \leq \nu} \log p.$$

Let p_i denote the $i-th$ prime.

We begin with some results on primes.

Lemma 3.1. *Let* $k \in \mathbb{Z}$ *and* $\nu \in \mathbb{R}$. *We have*

(i) $\pi(\nu) \geq \frac{\nu}{\log \nu - 1}$ *for* $\nu \geq 5393$ *and* $\pi(\nu) \leq \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$ *for* $\nu > 1$.
(ii) $\pi(\nu_1 + \nu_2) \leq \pi(\nu_1) + \pi(\nu_2)$ *for* $2 \leq \nu_1 < \nu_2 \leq \frac{7}{5}\nu_1(\log \nu_1)(\log\log \nu_1)$.
(iii) $\nu(1 - \frac{3.965}{\log^2 \nu}) \leq \theta(\nu) < 1.00008\nu$ *for* $\nu > 1$.
(iv) $p_k \geq k \log k$ for $k \geq 1$.
(v) $\text{ord}_p((k-1)!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ *for* $k \geq 2$.
(vi) $\sqrt{2\pi k}\ e^{-k}k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\ e^{-k}k^k e^{\frac{1}{12k}}$.

The estimates $(i), (ii)$ and (iii) are due to Dusart ([Dus99] and [Dus02], respectively). The estimate (iv) is due to Rosser [Ros38] and estimate (vi) is due to Robbins [Rob55, Theorem 6]. For a proof of (v), see [LaSh04, Lemma 2(i)]. □

We derive from Lemma 3.1 the following results.

Corollary 3.2. *Let $10^{10} < m \leq 123k$. Then there are primes p, q with $m \leq p < m+k$ and $\frac{m}{2} \leq q < \frac{m+k}{2}$.*

Proof. Let $10^{10} < m \leq 123k$. We observe that the assertion holds if

$$\theta(\frac{m+k-1}{s}) - \theta(\frac{m-1}{s}) = \sum_{\frac{m-1}{s} < p \leq \frac{m+k-1}{s}} \log p > 0$$

for $s = 1, 2$. Now from Lemma 3.1 and since $m > 10^{10}$, it suffices to show

$$\theta(\frac{m+k-1}{s}) - \theta(\frac{m-1}{s}) > \frac{m+k-1}{s}\left(1 - \frac{3.965}{\log^2(5\cdot 10^9)}\right) - 1.00008\frac{m-1}{s} > 0$$

or

$$k(1 - \frac{3.965}{\log^2(5\cdot 10^9)}) > (m-1)(\frac{8}{10^5} + \frac{3.965}{\log^2(5\cdot 10^9)}).$$

This is true since $m \leq 123k$ and

$$\frac{1 - \frac{3.965}{\log^2(5\cdot 10^9)}}{\frac{8}{10^5} + \frac{3.965}{\log^2(5\cdot 10^9)}} > 123.$$

□

Corollary 3.3. *We have*

$$\pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{6k}{5}) \leq \begin{cases} k-2 & \text{for } k \geq 61 \\ \pi(4k) & \text{for } k \geq 8000. \end{cases} \tag{14}$$

Proof. Let $k \geq 30000$. We have from $\frac{\log y}{\log x} = 1 + \frac{\log y/x}{\log x}$ and Lemma 3.1 (i) that

$$\begin{aligned} &(\log 4k)\left(\pi(4k) - \pi(\frac{6k}{5}) - \pi(k) - \pi(\frac{k}{2}) - \pi(\frac{k}{3}) - \pi(\frac{k}{4})\right) \\ \geq &\frac{4k}{\log 4k - 1} + \\ &k\left(4 - \frac{6}{5}\left(1 + \frac{\log\frac{10}{3}}{\log\frac{6k}{5}}\right)\left(1 + \frac{1.2762}{\log\frac{6k}{5}}\right) - \sum_{j=1}^{4}\frac{1}{j}\left(1 + \frac{\log 4j}{\log\frac{k}{j}}\right)\left(1 + \frac{1.2762}{\log\frac{k}{j}}\right)\right). \end{aligned}$$

The right hand side of the above inequality is an increasing function of k and it is positive at $k = 30000$. Therefore the left hand side of (14) is at most $\pi(4k)$ for $k \geq 30000$. By using exact values, we find that it is valid for $k \geq 8000$.

Also $\pi(4k) \leq \frac{4k}{\log 4k}\left(1 + \frac{1.2762}{\log 4k}\right) \leq k - 2$ is true for $k \geq 8000$. Therefore the left hand side of (14) is at most $k - 2$ for $k \geq 8000$. Finally we check using exact values of the $\pi-$function that the left hand side of (14) is at most $k - 2$ for $61 \leq k < 8000$. □

The following result is on Grimm's Conjecture, [LaSh06b, Theorem 1]. Grimm's Conjecture states that *given integers* $n \geq 1$ *and* $k \geq 1$ *such that whenever* $n + 1, \cdots, n + k$ *are all composite numbers, we can find distinct primes* P_i *with* $P_i|(n + i)$ *for* $1 \leq i \leq k$. This is a difficult conjecture having several interesting consequences. For example, this conjecture implies $p_{i+1} - p_i < p_i^{0.46}$ for sufficiently large i, a result better than that given by Riemann hypothesis. This follows by taking $n = p_i$ in [LaMu00, Theorem 1(i)]. We refer to [RST75] and [LaMu00] for a survey and results on Grimm's Conjecture.

Lemma 3.4. *Let* $m \leq 1.9 \cdot 10^{10}$ *and* $l \geq 1$ *be such that* $m+1, m+2, \cdots, m+l$ *are all composite numbers. Then there are distinct primes* P_i *such that* $P_i|(m + i)$ *for each* $1 \leq i \leq l$.

The following result follows from [SaSh03, Lemma 3].

Lemma 3.5. *Let* $m + k - 1 < k^{\frac{3}{2}}$. *Let* $|\{i : P(m + i) \leq k\}| = \mu$. *Then*

$$\binom{m+k-1}{k} \leq (2.83)^{k+\sqrt{m+k-1}}(m + k - 1)^{k-\mu}.$$

4. An upper bound for m when $\omega(\Delta(m, k)) \leq t$

Let m, k and t be positive integers such that

$$\omega(\Delta(m, k)) \leq t. \tag{15}$$

For every prime p dividing $\Delta(m, k)$, we delete a term $m + i_p$ in $\Delta(m, k)$ such that $\mathrm{ord}_p(m + i_p)$ is maximal. Then we have a set T of terms in $\Delta(m, k)$ with

$$|T| = k - t := t_0.$$

We arrange the elements of T as $m + i_1 < m + i_2 < \cdots < m + i_{t_0}$. Let

$$\mathfrak{P} := \prod_{\nu=1}^{t_0}(m + i_\nu) \geq m^{t_0}. \tag{16}$$

Now we obtain an upper bound for $\mathfrak{P}$. For a prime p, let r be the highest power of p such that $p^r \leq k-1$ and let i_0 be such that $\mathrm{ord}_p(m+i_0d)$ is maximal. Let $w_l = |\{m+i : p^l|(m+i), m+i \in$

$T\}|$ for $1 \le l \le r$. By an argument that was first given by Sylvester and Erdős(see []), we have $w_l \le [\frac{i_0}{p^l}] + [\frac{k-1-i_o}{p^l}] \le [\frac{k-1}{p^l}]$. Let $h_p > 0$ be such that $[\frac{k-1}{p^{h_p+1}}] \le t_0 < [\frac{k-1}{p^{h_p}}]$. Then there are at most $t_0 - w_{h_p+1}$ terms in T exactly divisible by p^l with $l \le h_p$. Hence

$$\begin{aligned}\mathrm{ord}_p(\mathfrak{P}) &\le rw_r + \sum_{u=h_p+1}^{r-1} u(w_u - w_{u+1}) + h_p(t_0 - w_{h_p+1}) \\ &= w_r + w_{r-1} + \cdots + w_{h_p+1} + h_p t_0 \\ &\le \sum_{u=1}^{r} \lfloor \frac{k-1}{p^u} \rfloor + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor = \mathrm{ord}_p((k-1)!) + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor.\end{aligned}$$

It is also easy to see that $\mathrm{ord}_p(\mathfrak{P}) \le \mathrm{ord}_p((k-1)!)$. Let $L_0(p) = \min(0, h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor)$. For any $l \ge 1$, we have from (16) that

$$m \le (\mathfrak{P})^{\frac{1}{t_0}} \le \left((k-1)! \prod_{p \le p_l} p^{L_0(p)}\right)^{\frac{1}{t_0}} =: L(k,l). \tag{17}$$

Observe that

$$m^{t_0} \le (L(k,l))^{t_0} \le (k-1)!. \tag{18}$$

5. Prelude to the proof of Theorems 3-5

Let $k \ge 2$, $n \ge 2k$, $a \ge 0, m = n + a - k + 1$ and $|a_0 a_n| = 1$. Then $m > k + a$. We consider the polynomials $f_{n,a}(x)$ with $3 < a \le 40$ when $k = 2$; $10 < a \le 50$ when $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \le \max(50, 5k)$ when $k \ge 5$. Let $P_1 > P_2 > \ldots > P_s \ge k + 2$ be primes dividing $\Delta(m, k)$. We write $P_{m,k} = \{P_1, P_2, \ldots, P_s\}$. We use Corollaries 2.1 and 2.2 to apply the following procedure which we refer to as *Procedure* $\mathcal{R}$.

Procedure $\mathcal{R}$: Let k be fixed. For all a with $3 < a \le 40$ if $k = 2$; $10 < a \le 50$ if $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \le \max(50, 5k)$ if $k \ge 5$, it suffices to consider only (m, k, a) with $P_1 \le k + a$ by Corollary 2.2 (i). We restrict to such triples (m, k, a) with $P_1 \le k + a$. By Corollary 2.2 (iii), we have $a \in \mathfrak{B}_0(m, k) := \mathfrak{B}\{P_1, P_2, \ldots, P_s\}$. Therefore we further restrict to (m, k, a) with $a \in \mathfrak{B}_0(m, k)$. Further for $k \in \{2, 3, 4, 5\}$ and $p = 5 \in P_{m,k}$ if $k = 2$; $p = 5 \in P_{m,k}$ or $p = 7 \in P_{m,k}$ if $k = 3$ and $p = 7 \in P_{m,k}$ if $k \in \{4, 5\}$, we restrict to those (m, k, a) with $a \notin \mathfrak{A}_{k,p}$ by using Corollary 2.1 and recalling $n = m + k - 1 - a$. Every (m, k, a) gives rise to the triplet (n, k, a).

We try to exclude the triplets (n, k, a) given by *Procedure* $\mathcal{R}$ to prove our theorems.

Let

$$\omega_0(a) = \begin{cases} \pi(a+k) & \text{if } a \le k+1 \\ \sum_{j=1}^{2}\left(\pi(\frac{a+k}{j}) - \pi(\max(k+1,\frac{a}{j}))\right) + \pi(k+1) & \text{if } k+1 < a \le 2k+2 \\ \sum_{j=1}^{3}\left(\pi(\frac{a+k}{j}) - \pi(\max(k+1,\frac{a}{j}))\right) + \pi(k+1) & \text{if } 2k+2 < a \le 3k+3 \\ \sum_{j=1}^{4}\left(\pi(\frac{a+k}{j}) - \pi(\max(k+1,\frac{a}{j}))\right) + \pi(k+1) & \text{if } 3k+3 < a \le 4k+4 \\ \sum_{j=1}^{5}\left(\pi(\frac{a+k}{j}) - \pi(\max(k+1,\frac{a}{j}))\right) + \pi(k+1) & \text{if } 4k+4 < a \le 5k \end{cases}$$

and ω_1 be the maximum of $\omega_0(a)$ for $1.5k < a \le 5k$. Then $\omega(\Delta(a+1,k)) \le \omega_1$.

Let $k \ge 10$. Assume that $\omega(\Delta(m,k)) > \omega_1$. Then there is a prime $p \ge k+2$ with $p|\Delta(m,k)$ such that $p \nmid \Delta(a+1,k)$ and $p \nmid a_0a_n$. Further $p \ge 13 > 2u_0$ since $u_0 \le 5$. Hence $f(x)$ has no factor of degree k by Lemma 1.1. Therefore we may suppose that

$$\omega(\Delta(m,k)) \le \omega_1 \text{ for } k \ge 10. \tag{19}$$

Let $k \ge 100$. Let $(i-1)(k+1) < a \le i(k+1)$ with $1 \le i \le 5$. For $1 \le j < i$, we have $\frac{a}{j} > \frac{k}{j} \ge \frac{100}{4}$ implying $\frac{\frac{a}{j}}{\frac{k}{j}} = \frac{a}{k} \le 5 \le \frac{7}{5}\log(25)\log\log(25) \le \frac{7}{5}\log(\frac{k}{j})\log\log(\frac{k}{j})$. Hence $\pi(\frac{a+k}{j}) - \pi(\frac{a}{j}) \le \pi(\frac{k}{j})$ for $1 \le j < i$ by Lemma 3.1 (ii). Therefore

$$\omega_0(a) \le \begin{cases} \pi(k+k+1) & \text{if } a \le k+1 \\ \pi(k) + \pi(\frac{k}{2}+k+1) & \text{if } k+1 < a \le 2k+2 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}+k+1) & \text{if } 2k+2 < a \le 3k+3 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}+k+1) & \text{if } 3k+3 < a \le 4k+4 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{k}{5}+k) & \text{if } 4k+4 < a \le 5k \end{cases}$$

which, again by Lemma 3.1 (ii), implies

$$\omega_1 \le \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{6k}{5}) =: \omega_2 \text{ for } k \ge 100. \tag{20}$$

Let $N_1(p) = \{N : P(N(N-1)) \le p\}$ and $N_2(p) = \{N : P(N(N-2)) \le p, N \text{ odd}\}$. Then N_1 and N_2 are given by [Leh64, Table IA] for $p \le 41$ and [Leh64, Table IIA] for $p \le 31$, respectively and we shall use them without reference. For given k, N and j with $1 \le j < k$, we put

$$M_j(N,k) = \prod_{i=0}^{k-1}(N-j+i).$$

Let

$$\mathcal{N}_j(k) := \{N \in N_1(41) : P(M_j(N,k)) \le 59\}.$$

By observing that

$$M_1(N, k+1) = M_1(N, k)(N-1+k),\ M_k(N, k+1) = (N-k)M_{k-1}(N, k)$$

and

$$M_j(N, k+1) = M_j(N, k)(N-j+k) = (N-j)M_{j-1}(N, k) \ \text{ for } 1 < j < k,$$

we can compute $\mathcal{N}_j(k)$ recursively as follows. Recall that $P(N(N-1)) \leq 41$ for $N \in N_1(41)$. Hence we have

$$\mathcal{N}_1(3) = \{N \in N_1(41) : P(N+1) \leq 59\},\ \mathcal{N}_2(3) = \{N \in N_1(41) : P(N-2) \leq 59\}.$$

For $k \geq 3$ and $1 \leq j \leq k$, we obtain $\mathcal{N}_j(k+1)$ recursively by

$$\mathcal{N}_1(k+1) = \{N \in \mathcal{N}_1(k) : P(N-1+k) \leq 59\},\ \mathcal{N}_k(k+1) = \{N \in \mathcal{N}_{k-1}(k) : P(N-k) \leq 59\}$$

and

$$\mathcal{N}_j(k+1) = \{N \in \mathcal{N}_j(k) : P(N-j+k) \leq 59\} \cup \{N \in \mathcal{N}_{j-1}(k) : P(N-j) \leq 59\} \text{ for } 1 < j < k.$$

6. Proof of Theorems 3 and 4 for $k < 10$

Let $k = 2$. Then $a \leq 40$. By Corollary 2.2 (i), we first restrict to those m for which $P(m(m+1)) \leq 41$. They are given by $m = N-1$ with $N \in N_1(41)$. By *Procedure* $\mathcal{R}$, we obtain the tuples $(n, 2, a)$ given in the following table.

a	$n+a$	a	$n+a$	a	$n+a$
4, 5	9	4	10	5, 6	28, 49, 64
4, 8, 9	16, 25, 81	9	33, 45, 55, 100, 121, 243	10	33, 243
12	27, 28, 49, 64, 91, 169, 729	13	21, 25, 28, 36, 50, 64	14	25
13, 14 19, 33	81, 126, 225, 2401, 4375	15, 16	289	17	513
18	25, 76, 81, 96, 361, 513, 1216	19	25, 28, 36, 49, 50, 64, 243	20	28, 33, 49, 64, 243
21	25, 33, 45, 55, 529	21, 22	46, 81, 100, 121, 576	23	81
24	40, 81, 65, 325, 625, 676	26	49, 64	27	49, 64, 784
28	81, 145	29	81, 125, 961	31	243
32	243, 289, 1089	33	49, 50, 51, 64, 85, 136, 256, 289, 5832	34	49, 50, 64, 81
36	1369	38	65, 81, 325, 625, 676	39	81, 82, 1025, 6561
40	49, 64, 82, 288				

Let $3 \le k \le 9$. Then $10 < a \le 50$ if $k = 3, 4$ and $30 < a \le 50$ if $5 \le k \le 9$. Thus we may assume that $P(\Delta(m,k)) \le 59$ by Corollary 2.2 (i).

Let $m \le 10000$. We need to consider $[k, 59] \cup \mathcal{M}(k)$ where $\mathcal{M}(k) = \{60 \le m \le 10000 : P(\Delta(m,k)) \le 59\}$. We compute $\mathcal{M}(3)$ and further from the identity $\Delta(m, k+1) = (m+k)\Delta(m,k)$, we obtain $\mathcal{M}(k+1) = \{m \in \mathcal{M}(k) : P(m+k) \le 59\}$ for $k \ge 3$ recursively. In fact we get

$$\mathcal{M}(6) = \{90, 91, 116, 184, 185, 285, 340\}, \quad \mathcal{M}(7) = \{90, 184\}$$

and $\mathcal{M}(8) = \mathcal{M}(9) = \emptyset$. We now apply Procedure $\mathcal{R}$ on $m \in [k, 59] \cup \mathcal{M}(k)$. We get

a	$n+a$	a	$n+a$
11	28	12	26, 27, 28, 65
19, 20	56, 100	20	46, 162
21	46	32	51, 56, 100, 121
33	51	38, 39	82
41, 43	56, 100	43, 44, 45	162

or $a \in \{12, 13, 18, 19, 20, 27, 32, 33, 34, 39, 41, 43, 44\}$, $n + a = 50$ if $k = 3$ and

a	$n+a$	a	$n+a$	a	$n+a$	a	$n+a$
11, 12	27, 28	13, 31, 32, 33	51	18	57	10	66

if $k = 4$.

Thus $m > 10000$. Suppose that $m + j = N \in N_1(41)$ for some $1 \le j < k$. Then $\Delta(m,k) = M_j(N,k)$ which implies $N \in \mathcal{N}_j(k)$ since $P(\Delta(m,k)) \le 59$. Let $\mathcal{N}'_j(k) = \{m \in \mathcal{N}_j(k) : m > 10000\}$. We find that

$\mathcal{N}'_1(3) = \{13311, 13455, 17576, 17577, 19551, 29601, 32799, 212381\}$

$\mathcal{N}'_2(3) = \{10881, 11662, 13312, 13456, 13690, 16170, 17577, 23375, 27456, 31213, 134850, 212382, 1205646\}$

$\mathcal{N}'_1(4) = \{17576\}$, $\mathcal{N}'_2(4) = \{17577\}$, $\mathcal{N}'_3(4) = \{10881\}$

and $\mathcal{N}'_j(k) = \emptyset$ for $k \ge 5$ and $1 \le j < k$. We now take $m = N - j$ with $N \in \mathcal{N}_j(k)$ for $1 \le j < k$ and apply Procedure $\mathcal{R}$ to find that there are no triplets (n, k, a).

Thus we may suppose that $m + j \notin N_1(41)$ for all $1 \le j < k$. Then $P((m+i)(m+i+1)) > 41$ for each $0 \le i < k - 1$. By Corollary 2.2 (i), we may suppose that $P(\Delta(m,k)) \le 53$ for $k \le 8$ and $P(\Delta(m,k)) \le 59$ for $k = 9$. Taking $V(m,k) = \{P((m+2i)(m+2i+1)) : 0 \le i < \frac{k}{2}\}$, we have $V(m,k) \subseteq \{43, 47, 53\}$ for $4 \le k \le 7$ and $V(m,k) = \{43, 47, 53, 59\}$ if $k = 8, 9$. Then $k \ne 8$ and computing $\{a \le 50 : a \in \mathfrak{B}\{Q_1, Q_2\}$ for $(Q_1, Q_2) \in \{(47, 43), (53, 43, (53, 53)\}\}$ if $k = 4, 5$;

$(Q_1, Q_2) = (53, 43)$ if $k = 6, 7, 9$, we find that the set is empty except when $k = 5, (Q_1, Q_2) = (43, 47)$ where it is $\{42\}$. Thus we may assume that $k = 5$ and $a = 42$. Further $P(\Delta(m, k)) = 47$ and $43|\Delta(m, k)$. If $p|\Delta(m, k)$ with $13 \leq p \leq 41$, then $42 \notin \mathfrak{B}\{47, p\}$ by Corollary 2.2 (*iii*). Thus we may further suppose that $p|\Delta(m, k)$ with $p \leq 11$ or $p \in \{43, 47\}$. Also $P(m) \leq 41$ otherwise each of $P(m), P((m+1)(m+2)), P((m+3)(m+4))$ is > 41 which is not possible. Again we get $P(m+2) \leq 41$ since otherwise each of $P(m(m+1)), P(m+2), P((m+3)(m+4))$ is > 41. Therefore $P(m(m+2)) \leq 41$ implying $P(m(m+2)) \leq 11$. If m is odd, then $m = N-2$ for $N \in N_2(11)$ and we check that there is a prime $p > 11, p \notin \{43, 47\}$ with $p|\Delta(m, k)$ which is a contradiction. Thus m is even and we have $P(\frac{m}{2}(\frac{m}{2}+1)) \leq 11$ implying $m = 2N - 2$ with $N \in N_1(11)$. This is again not possible as above.

Let $k = 3$. Then $P(\Delta(m, k)) \leq 53$ by Corollary 2.2 (*i*). Recall that $P_1 > P_2 > \cdots \geq k+2$ are all the primes dividing $\Delta(m, k)$. We observe that $P_1 > 41$ since $m + j \notin N_1(41)$ for $1 \leq j < k$. Further $P((m+1)(m+2)) > 41$ if $P(m) > 41$ and $P(m(m+1)) > 41$ if $P(m+2) > 41$ which are excluded by Corollary 2.2 (*iii*) as above. Thus we may suppose that $P_1 = P(m+1) > 41$ and $P(m(m+2)) \leq 41$. If m is even, then $m = 2N - 2$ for $N \in N_1(41)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \ldots\}$. Thus m is odd. If $P(m(m+2)) \leq 31$, then $m = N-2$ with $N \in N_2(31)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \ldots\}$ which is excluded. Thus $P_2 = P(m(m+2)) \in \{37, 41\}$ which together with $41 < P_1 \leq 53$ implies $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2\}$ except when $P_1 = 43, P_2 = 41$ where $a = 40 \in \mathfrak{B}\{P_1, P_2\}$. Thus $a = 40, P(m+1) = 43$ and $P(m(m+2)) = 41$. Further by Corollary 2.2 (*iii*), we may assume $p \in \{2, 3, 7, 41, 43\}$ for $p|\Delta(m, 3)$ and $2 \cdot 43|(m+1)$. By looking at the possible prime factorisations of $m, m+1, m+2$ and taking $(m+2)-m$ or $m-(m+2)$, we have the following possibilities.

$$\begin{aligned}
m+1 &= 2^r \cdot 7^y \cdot 43^t, & 3^x - 41^z &= \pm 2;\\
m+1 &= 2^r \cdot 3^x \cdot 43^t, & 7^y - 41^z &= \pm 2;\\
m+1 &= 2^r \cdot 43^t, & 3^x - 41^z &= \pm 2;\\
m+1 &= 2^r \cdot 43^t, & 3^x \cdot 7^y - 41^z &= \pm 2;\\
m+1 &= 2^r \cdot 43^t, & 3^x - 7^y \cdot 41^z &= \pm 2;\\
m+1 &= 2^r \cdot 43^t, & 7^y - 3^x \cdot 41^z &= \pm 2;
\end{aligned}$$

where r, x, y, z, t are positive integers. The second and fourth equations are excluded by taking remainders modulo 7. Calculating modulo 8 for the remaining possibilities, we get the following four simultaneuos equations.

$C1:$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 7^y \cdot 43^t = 1,$	$2^r \cdot 7^y \cdot 43^t - 41^z = 1,\ x$ odd
$C2:$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 41^z = 1,\ x$ odd
$C3:$	$3^x - 7^y \cdot 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y \cdot 41^z = 1$
$C4:$	$3^x \cdot 41^z - 7^y = 2,$	$3^x \cdot 41^z - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y = 1$

If $4|2^r$ in $C2$, we get a contradiction by taking remainders modulo 4 since x is odd, thus $2^r = 2$. Calculating modulo 7 in all the possibilities, we find that $C1$ is excluded since x is odd. Further $6|(x-1)$ in $C2$; $6|(x-2)$, $3|r$ in $C3$ and $3|r$ in $C4$. Note that $x \geq 2$. Taking remainders modulo 9 again, we find that $3|(z+1)$ in $C2$; $3|t$ in $C3$ and $3|t, 3|(y-1)$ in $C4$. Thus we have $(-41^{\frac{z+1}{3}})^3 + 3\cdot 41(3^{\frac{x-1}{3}})^3 = 2\cdot 41$ in $C2$, $(-2^{\frac{r}{3}}\cdot 43^{\frac{t}{3}})^3 + 9(3^{\frac{x-2}{3}})^3 = 1$ in $C3$ and $(2^{\frac{r}{3}}\cdot 43^{\frac{t}{3}})^3 + 7(-7^{\frac{y-1}{3}})^3 = 1$ in $C4$. We solve the Thue equations $X^3+123Y^3 = 82, X^3+9Y^3 = 1$ and $X^3+7Y^3 = 1$ with X, Y integers in **PariGp** to find that it is not possible.

We recall that Theorem 4 follows from Theorem 3 when $k \geq 10$. Therefore we prove Theorem 3 with $k \geq 10$ in Sections 7, 8 and this will complete the proofs of Theorems 3 and 4.

7. Proof of Theorem 3 for $k \geq 10$

We may suppose by Corollary 2.2 (i) that $P(\Delta(m,k)) \leq a+k \leq 6k$. Let $k \leq 17$. We may suppose that $\max(30, 1.5k) < a \leq 5k$. First assume that $m+j \notin N_1(41)$ for any $1 \leq j < k$. Let

$$\mathfrak{L}_i(k,a) := \{p : \max(41, \frac{a}{i}) < p \leq \frac{a+k}{i}\} \ \text{ for } \ 1 \leq i \leq 5$$

and $\ell(k) := \max_{1.5k<a\leq 5k} |\cup_{i=1}^{5} \mathfrak{L}_i(k,a)|$. There are at most $\ell(k)$ primes > 41 dividing $\Delta(a+1,k)$ and we delete numbers in $\{m, m+1, \cdots, m+k-1\}$ divisible by those primes. We are left with at least $k-\ell(k)$ numbers. We observe that the prime factors of each of these numbers are at most 41 otherwise the assertion follows by Lemma 1.1. We call U the largest such number. From [Leh64, Tables IA], we may assume that each of these numbers is at least at a distance 2 from the preceding one. Thus $m+k-1 \geq U \geq m+2(k-\ell(k)-1)$. Hence we have a contradiction if $k-2\ell(k)-1 > 0$. This is the case since $\ell(k) = 2,3,4,5$ when $k = 10, k \in \{11,12\}$, $k \in \{13,14\}$, $k \in \{15,16,17\}$, respectively. Therefore we suppose that $m+j_0 = N \in N_1(41)$ for some $1 \leq j_0 \leq k-1$. Then $\Delta(m,k) = M_{j_0}(N,k)$. We check that $P(M_j(N,7)) > 102$ for $1 \leq j < 7$ when $N > 10000$ and $N \in N_1(41)$. Thus $m < N \leq 10000$. For each $m < 10000$, we check that $P(\Delta(m,10)) > 102$ for $m \geq 118$. Therefore $P(\Delta(m,k)) > 6k$ when $m \geq 118$. Further we find that $p_{i+1} - p_i \leq 10$ for $p_i < 118$. Hence for $m < 118$, $P(\Delta(m,k)) \geq m$ since $k \geq 10$. Therefore we have $P(\Delta(m,k)) \geq \min(m, 6k+1) > k+a$ for all m. Now the assertion follows by Corollary 2.2 (i).

Thus $k \geq 18$. First we check that $\omega_1 < k$ for $k \leq 100$ which together with (20) and Corollary 3.3 implies $\omega_1 < k$ for all k. Suppose $m \leq 10^{10}$. If at least one of $m, m+1, \ldots, m+k-1$ is a prime, then $P(\Delta(m,k)) \geq m > k+a$ and therefore the assertion follows from Corollary 2.2 (i). Hence we may suppose that each of $m, m+1, \ldots, m+k-1$ is composite. By Lemma 3.4, we obtain $\omega(\Delta(m,k)) \geq k > \omega_1$ which contradicts (19). Therefore we have $m > 10^{10}$ which implies $k > 500$ by (19) and (17) with $t_0 = \omega_1$.

By (19) and (20), we have $\omega(\Delta(m,k)) \leq \omega_2$. We obtain from (18), Lemma 3.1 (vi) and $k > 500$ that

$$m^{k-\omega_2} < (k-1)! = \frac{k!}{k} \leq \frac{\sqrt{2\pi k}}{k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}} < \left(\frac{k}{e}\right)^k. \tag{21}$$

Since $m \geq 10^{10}$, we get

$$\log k - 1 > \frac{(k-\omega_2)\log m}{k} \geq 10(\log 10)(1 - \frac{\omega_2}{k}).$$

By using estimates of $\pi(\nu)$ from Lemma 3.1 (i), we obtain

$$k > e^{\left(1+10(\log 10)\left(1-\frac{\frac{6}{5}}{\log\frac{6k}{5}}\left(1+\frac{1.2762}{\log\frac{6k}{5}}\right)-\sum_{j=1}^{4}\frac{1}{j\log\frac{k}{j}}\left(1+\frac{1.2762}{\log\frac{k}{j}}\right)\right)\right)} =: J(k)$$

Since $J(k)$ is an increasing function of k and $k > 500$, we have $k > J(500) \geq 4581$. Further $k > J(4581) \geq 578802$ and hence $k > J(578802) > 4.5 \times 10^7$. Let $m \leq 123k$. Then, by Corollary 3.2, there is a prime $P_1 \geq m$ such that $P_1 | \Delta(m,k)$. Since $m > a+k$, the assertion follows by Corollary 2.2 (i). Therefore we may suppose that $m > 123k$.

Assume that $m + k - 1 \geq k^{\frac{3}{2}}$. Then $m > \frac{k^{\frac{3}{2}}}{e}$ and we get from (21) and Corollary 3.3 that

$$k^k > (k^{\frac{3}{2}})^{k-\pi(4k)}$$

which together with estimates of $\pi(\nu)$ from Lemma 3.1 implies

$$0 > \frac{k - 3\pi(4k)}{k} \geq 1 - \frac{12}{\log 4k}\left(1 + \frac{1.2762}{\log 4k}\right).$$

The right hand expression is an increasing function of k and the inequality does not hold at $k = 10^6$. Therefore $m + k - 1 < k^{\frac{3}{2}}$. By Lemma 3.5, we get

$$\binom{m+k-1}{k} \leq (2.83)^{k+k^{\frac{3}{4}}} k^{\frac{3}{2}(\pi(4k)-\pi(k))}$$

since $|\{i : P(m+i) \leq k\}| \geq k - (\pi(4k) - \pi(k))$ by (15) and Corollary 3.3. On the other hand, we have $m > 123k$ implying

$$\binom{m+k-1}{k} \geq \binom{124k}{k} = \frac{(124k)!}{k!(123k)!} > \frac{\sqrt{2\pi(124k)}(\frac{124k}{e})^{124k}}{\sqrt{2\pi k}(\frac{k}{e})^k e^{\frac{1}{12k}}\sqrt{2\pi(123k)}(\frac{123k}{e})^{123k} e^{\frac{1}{12\cdot 123k}}}$$

$$> \frac{0.4}{\sqrt{k}} e^{-\frac{1}{8k}}(335.7)^k$$

using estimates of $\nu!$ from Lemma 3.1. Comparing the upper and lower bounds, we obtain

$$0 > \log(0.4) - \frac{1}{8k} - 0.5\log k + k\log(\frac{335.7}{2.83}) - k^{\frac{3}{4}}\log(2.83) - \frac{3}{2}(\pi(4k) - \pi(k))\log k.$$

By using estimates of $\pi(\nu)$ from Lemma 3.1 again, we obtain

$$\begin{aligned}\frac{(\pi(4k)-\pi(k))\log k}{k} &\le \frac{4\log k}{\log 4k}(1+\frac{1.2762}{\log 4k})-\frac{\log k}{\log k-1}\\ &\le 4\left(1-\frac{\log 4}{\log 4k}\right)\left(1+\frac{1.2762}{\log 4k}\right)-1\\ &\le 4\left(1-\frac{\log 4-1.2762}{\log 4k}\right)-1\le 3.\end{aligned}$$

Therefore we have

$$0>\frac{\log(0.4)-\frac{1}{8k}-0.5\log k}{k}+\log(\frac{335.7}{2.83})-k^{-\frac{1}{4}}\log(2.83)-4.5.$$

The right hand side of the above inequality is an increasing function of k and the inequality is not valid at $k=10^6$. This is a contradiction. □

8. Proof of Theorem 5

By Theorem 4, we restrict to those triplets (n,a,k) given in the statement of Theorem 4 with $a\le 12$. We now factorize $f_{n,a}(x)$ with $a_0a_n=\pm 1, a_1=a_2=\ldots=a_{n-1}=1$ to find that these $f_{n,a}(x)$ are irreducible. Hence the assertion follows. □

9. Proof of Theorem 1

For the proof of Theorem 1, we put $\alpha=a$ throughout this section. As remarked in Section 1 after the statement of Theorem 1, we may assume that $10<a\le 40$. For $n\le 18$ and $n\in\{24,25,27,30,32,36,45,48,54,60,64,72,75,80,90,112,120\}$, we find that $L_n^{(a)}(x)$ is irreducible except for (n,a) listed in Theorem 1. Thus we assume $n>18$, $n\notin\{24,25,27,30,32,36,45,48,54,60,64,72,75,80,90,112,120\}$. Assume that $L_n^{(\alpha)}(x)$ is reducible. Then $L_n^{(\alpha)}(x)$ has a factor of degree k with $1\le k\le\frac{n}{2}$. First we prove the following lemma.

Lemma 9.1. *Let $k\ge 2$. Then $L_n^{(a)}(x)$ has no factor of degree k.*

Proof. Let $k\ge 2$ and $a\le 40$ if $k=2$. We may restrict to those (n,k,a) given in the list of exceptions in Theorem 4. For each of these triplets (n,k,a), we first check if there is a prime $p\ge k+2$ with (10) such that either (8) or (9) is satisfied and they can be excluded by Lemma 1.1. We are now left with triples (n,k,a) given by $k=2,(n,a)\in\{(100,21),(40,24),(256,33),(42,40)\}$. For these (n,a), we check that $L_n^{(a)}(x)$ is irreducible.

Let $k=2$ and $40<a\le 50$. Suppose $n\notin N_1(23)$ and $n+a\notin N_1(23)$. Then $P_1=P(n(n-1))>23$ and $P_2=P((n+a)(n+a-1))>23$. Further either $P_1\nmid(a+1)(a+2)$ or

$P_2 \nmid (a+1)(a+2)$ and then the assertion follows by Lemma 1.1. Therefore we may assume that either $n = N \in N_1(23)$ or $n + a = N \in N_1(23)$. Further we may also suppose that $P(n(n-1)(n+a)(n+a-1)) \le P((a+1)(a+2))$ since otherwise the assertion follows by Lemma 1.1. For $N \in N_1(23)$ and $N > 10000$, we check that $P((N-a)(N-a-1)) > P((a+1)(a+2))$ and $P((N+a)(N+a-1)) > P((a+1)(a+2))$ except when $(a, N) \in \{(45, 10648), (46, 12168)\}$ where $P(N(N-1)) \in \{13, 23\}$, respectively. Observe that $N(N-1)|n(n-1)(n+a)(n+a-1)$. By taking $p = P(N(N-1))$, the assertion follows from Lemma 1.1. We now consider $n \le 10000$. Let a be given. By Lemma 1.1, we first restrict to those n for which $P(n(n-1)(n+a)(n+a-1)) \le P((a+1)(a+2))$. Further we check that there is a prime $p|n(n-1)(n+a)(n+a-1), p > 7$ and $p \nmid (a+1)(a+2)$. Lemma 1.1 implies the assertion now. □

By Lemma 9.1, we only need to consider $k = 1$. If there is a prime $p|n(n+a), p \nmid (a+1)$ with either $p \ge 11$ or $p = 7, a \ne 47$ or $p = 5, a \notin \{23, 48\}$ or $p = 3, a \notin \{16, 24, 25, 34, 43\} =: S_1$, then the assertion follows by Lemma 1.1 and Corollary 2.1. Let $P_a = \{2\} \cup \{p : p|(a+1)\}$ if $a \notin S_1 \cup \{23, 47, 48\}$, $P_a = \{2, 3\} \cup \{p : p|(a+1)\}$ if $a \in S_1$, $P_a = \{2, 3, 5\}$ if $a = 23$, $P_a = \{2, 3, 7\}$ if $a = 47$ and $P_a = \{2, 5, 7\}$ if $a = 48$. Thus for a given a, we may assume that $p|n(n+a)$ implies $p \in P_a$.

Let a be given. Let $p|n$ with $p > 2$. Then $p \in P_a$. As in the proof of Lemma 1.1, if we have $\phi'_j < 1$ for all $1 \le j \le n$, then $L_n^{(\alpha)}(x)$ does not have a linear factor and we are done. Let $1 \le j \le 50$. We compute ϕ_j to find that $\phi_j < 1$ for $j > 1$ except when $(p, a) \in T_1 := \{(3, 16), (3, 17), (3, 34), (3, 35), (3, 43), (3, 44), (5, 23), (5, 24), (5, 48), (5, 49), (7, 47), (7, 48)\}$ where $\phi_j < 1$ for $j > 2$ and except when $23 \le a \le 26, p = 3$ where $\phi_j < 1$ for $j > 4$. Let $j > 50$. By using $\text{ord}_p(s!) \le \frac{s}{p-1}$, we find that

$$\phi_j = \frac{\text{ord}_p((a+j)!) - \text{ord}_p(a!)}{j} \le \frac{\frac{a+j}{p-1} - \text{ord}_p(a!)}{j} \le \frac{1}{p-1} + \frac{\frac{a}{p-1} - \text{ord}_p(a!)}{51} < 1.$$

It suffices to show that $\phi'_1 < 1$ except when $(p, a) \in T_1$ for which we need to show $\phi'_j < 1$, $1 \le j \le 2$ and except when $23 \le a \le 26, p = 3$ for which we need to show $\phi'_j < 1$ for $1 \le j \le 4$. Let $\phi'_0 = \max\{\phi'_i\}$ for $1 \le i \le 4$. It suffices to show $\phi'_0 < 1$ is always valid. This is the case except when $a \in \{24, 49\}, p = 5$; $a \in \{17, 24, 25, 26, 35, 44\}, p = 3$ and $a = 48, p = 7$. Further $\text{ord}_5(n) \le 1$ when $a \in \{24, 49\}$, $\text{ord}_7(n) \le 1$ when $a = 48$, $\text{ord}_3(n) \le 1$ when $a \in \{17, 24, 25, 35, 44\}$ and $\text{ord}_3(n) \le 2$ when $a = 26$ otherwise $\phi'_0 < 1$. Let $a \in \{17, 26, 35\}$ and $\text{ord}_3(n) = 1$ or $\text{ord}_3(n) = 2$. Then from $n(n+a) = 2^\alpha 3^{\beta_3}$ and $gcd(n, n+a) \le 2$, we obtain $n \in \{3, 6, 9, 18\}$ which is not possible. Let $a = 49$ and $\text{ord}_5(n) = 1$. Then from $n(n+a) = 2^\alpha 5^{\beta_5}$ and $\gcd(n, n+a) = 1$, we obtain $n = 5$ which is again not possible. Here $\gcd(a, b)$ stands for greatest common divisor of a and b.

Therefore n is a power of 2 except when $a = 24$ where $\mathrm{ord}_3(n) \le 1$ or $\mathrm{ord}_5(n) \le 1$; $a = 25$ where $\mathrm{ord}_3(n) \le 1$; $a = 44$ where $\mathrm{ord}_3(n) \le 1$ and $a = 48$ where $\mathrm{ord}_7(n) \le 1$. From the definition of P_a, we observe that $n(n+a)$ has at most two odd prime factors except when $a = 34$ where it has at most three odd prime factors. Hence we always have $n, n+a$ of the form

$$\text{(22)} \qquad \begin{array}{lll} n = 2^{\alpha+\delta},\ \frac{n+a}{2^\delta} = & p^{\beta_p} & \text{if } P_a = \{2, p\} \\ n = 2^{\alpha+\delta},\ \frac{n+a}{2^\delta} \in & \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}\} & \text{if } P_a = \{2, p_1, p_2\} \\ n = 2^{\alpha+\delta},\ \frac{n+a}{2^\delta} \in & \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_3^{\beta_{p_3}}, & \\ & p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}\} & \text{if } P_a = \{2, p_1, p_2, p_3\}. \end{array}$$

where $2^\delta || a$ and in addition $n, n+a$ is of the form

$$\text{(23)} \qquad \begin{array}{lll} n = 15 \cdot 2^{\alpha+3}, & n + a = 8 \cdot 3^{\beta_3+1} \quad \text{or} & \\ n = 3 \cdot 2^{\alpha+3}, & n + a \in \{8 \cdot 3^{\beta_3+1}, 8 \cdot 3^{\beta_3+1} 5^{\beta_5}\} & \text{if } a = 24 \\ n = 3 \cdot 2^{\alpha}, & n + a = 13^{\beta_{13}} & \text{if } a = 25 \\ n = 3 \cdot 2^{\alpha+2}, & n + a = 4 \cdot 5^{\beta_5} & \text{if } a = 44 \\ n = 7 \cdot 2^{\alpha+4}, & n + a = 16 \cdot 5^{\beta_5} & \text{if } a = 48. \end{array}$$

Here all the exponents of odd prime powers appearing in (22) and (23) are positive. For $n < 512$ and n of the form given by (22) or (23) which are given by $n \in \{96, 128, 192, 224, 240, 256, 384, 448, 480\}$, we check that there is a prime $p|(n+a), p \notin P_a$ except when $(n, a) \in \{(256, 14), (128, 16), (256, 16), (96, 24), (192, 24), (256, 32), (256, 33), (128, 34)\}$. We find that for each of these (n, a), the polynomial $L_n^{(a)}(x)$ is irreducible. Therefore we have $n \ge 512$.

From the equality $\frac{n+a}{2^\delta} - \frac{n}{2^\delta} = \frac{a}{2^\delta}$, we obtain an equation of the form

$$p^{\beta_p} - 2^\alpha = \frac{a}{2^\delta} \quad \text{or} \quad p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} - 2^\alpha = \frac{a}{2^\delta}$$

or further $3^{\beta_3} 5^{\beta_5} 7^{\beta_7} - 2^\alpha = 17$ (only when $a = 34$) or $3^{\beta_3} - 5 \cdot 2^\alpha = 1$ (only when $a = 24$) or $13^{\beta_{13}} - 3 \cdot 2^\alpha = 25$ (only when $a = 25$) or $5^{\beta_5} - 3 \cdot 2^\alpha = 11$ (only when $a = 44$) or $5^{\beta_5} - 7 \cdot 2^\alpha = 3$ (only when $a = 48$). In each of the equations thus obtained, we note that $8 | 2^\alpha$ since $n \ge 512$. Out of all the equations, we need to consider only those which are valid under remainders modulo 8 and hence we restrict to those. Here we use $p^{\beta_p} \equiv 1$ or p modulo 8 according as β_p is even or odd, respectively. They are now expressed as the Thue equation

$$X^3 + AY^3 = B$$

and we solve them in **PariGp**. For instance, let $a = 32$. Then we obtain equations of the form $3^{\beta_3} - 2^\alpha = 1$, $11^{\beta_{11}} - 2^\alpha = 1$, $3^{\beta_3} 11^{\beta_{11}} - 2^\alpha = 1$. By taking remainders modulo 8, we find that $\beta_3, \beta_{11}, \beta_3 + \beta_{11}$ are even for the first, second and third equation, respectively. This implies $3^{\frac{\beta_3}{2}} - 1 = 2, 3^{\frac{\beta_3}{2}} + 1 = 2^{\alpha-1}$ giving $3^{\beta_3} = 9, 2^\alpha = 8$ for the first equation and $11^{\frac{\beta_{11}}{2}} - 1 = 2, 11^{\frac{\beta_{11}}{2}} + 1 = 2^{\alpha-1}$ giving a contradiction for the second equation. Observe that $2^\alpha > 8$ since $n \ge 512$. Thus we are left with $3^{\beta_3} 11^{\beta_{11}} - 2^\alpha = 1$. For some $0 \le r, s, t \le 2$, we have $\alpha+r, \beta_3-s, \beta_{11}-t$ all are multiples of 3 and from $-2^{\alpha+r}+2^r3^s11^t3^{\beta_3-s}11^{\beta_{11}-t}=2^r$, we obtain the Thue equations $X^3+AY^3=B$ with $B=2^r, A=2^r3^s11^t, 0\le r,s,t\le 2$ and with X a power of 2 and $33|AY$. There are 27 possibilities of pairs (A,B). If $A=1$, then $B=1$ and we factorise X^3+Y^3 to get a contradiction. Thus the case $A=1$ is excluded. For all other values of (A,B) than those given by $t=2$, we check in **PariGp** that none of the solutions (X,Y) of Thue equations thus obtained satisfy the condition X a power of 2 and $33|AY$ except when $A=66, B=2$ where $X=-4$ and $Y=1$ from which we obtain $n=1024$. When $t=2$, from $3^{\beta_3-s+3}11^{\beta_{11}-2+3}-2^{3-r}3^{3-s}\cdot 11\cdot 2^{\alpha+r-3}=3^{3-s}\cdot 11$, we obtain the Thue equations $X^3+AY^3=B$ with $B=3^{3-s}\cdot 11, A=2^{3-r}3^{3-s}\cdot 11, 0\le r,s\le 2$ and $33|X$ and Y a power of 2. We check again in **PariGp** that none of the solutions (X,Y) of these Thue equations thus satisfy the condition $33|X$ and Y a power of 2. Hence we need to consider $n=1024$ when $a=32$. For another example, let $a=48$. We obtain equations of the form $5^{\beta_5}-2^{\alpha}=3$, $7^{\beta_7}-2^{\alpha}=3$, $5^{\beta_5}-7\cdot 2^{\alpha}=3$ and $5^{\beta_5}7^{\beta_7}-2^{\alpha}=3$. The first three equations are excluded modulo 8 and for the last equation, we find that β_5,β_7 are both odd. Taking remainders modulo 7 imply $3|(\alpha-2)$ or $3|(\alpha+1)$ and hence from the equation $-2^{\alpha+1}+2\cdot 5^{\beta_5}7^{\beta_7}=6$, we obtain the Thue equations $X^3+AY^3=B$ with $B=6, A=2\cdot 5^s7^t, 0\le s,t\le 2$ and X a power of 2 and $70|AY$. When $t=2$, from $5^{\beta_5-s+3}7^{\beta_7+1}-4\cdot 5^{3-s}\cdot 7\cdot 2^{\alpha-2}=3\cdot 5^{3-s}\cdot 7$, we obtain the Thue equations $X^3+AY^3=B$ with $B=21\cdot 5^{3-s}, A=28\cdot 5^{3-s}, 0\le s\le 2$ and $35|X$ and Y a power of 2. We check in **PariGp** that all the solutions (X,Y) of these Thue equations are excluded except when $(A,B)=(70,6)$ where $X=-4, Y=-1$ and we obtain $n=512$. Hence we need to consider $n=512$ when $a=48$. Similarly, all other a's are excluded except when $a\in\{20,24\}$ where we obtain $(n,a)\in\{(4096,20),(1920,24)\}$.

Thus we now exclude the cases $(n,a)\in\{(4096,20),(1920,24),(1024,32),(512,48)\}$. We take $p=2$ and show that $\phi'_j<1$ for all $1\le j\le n$. This is shown by checking $\mathrm{ord}_2(\Delta_j)-\mathrm{ord}_2(\binom{n}{j})<j$ for j such that $\mathrm{ord}_2(\Delta_j)\ge j$ for these pairs (n,a). Hence they are all excluded. □

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