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Asymptotic normality of Hill Estimator for truncated data

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ASYMPTOTIC NORMALITY OF HILL ESTIMATOR FOR TRUNCATED DATA

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Abstract. The problem of estimating the tail index from truncated data is addressed in **?**. In that paper, a sample based (and hence random) choice of k is suggested, and it is shown that the choice leads to a consistent estimator of the inverse of the tail index. In this paper, the second order behavior of the Hill estimator with that choice of k is studied, under some additional assumptions. In the untruncated situation, it is well known that asymptotic normality of the Hill estimator follows from the assumption of second order regular variation of the underlying distribution. Motivated by this, we show the same in the truncated case in light of the second order regular variation.

1. Introduction

Historically, one of the most important statistical issues related to distributions with regularly varying tail is estimating the tail index. A detailed discussion on estimators of the tail index can be found in Chapter 4 of **?**. One of the most popular estimators is the Hill estimator, introduced by **?**. For a one-dimensional non-negative sample $X_1, \ldots, X_n$, the Hill statistic is defined as

$$h(k,n) := \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k)}}, \tag{1.1}$$

where $X_{(1)} \geq \ldots \geq X_{(n)}$ are the order statistics of $X_1, \ldots, X_n$, and $1 \leq k \leq n$ is an user determined parameter. It is well known that if $X_1, \ldots, X_n$ are a i.i.d. sample from a distribution whose tail is regularly varying with index $-\alpha$ and k satisfies $1 \ll k \ll n$, then $h(k,n)$ consistently estimates α^{-1}. In a sense made precise by **?**, the consistency of Hill statistic is equivalent to the regular variation of the tail of the underlying distribution. Various authors have studied the second order behavior of the Hill estimator; see for example **?**, **?**, **?**, **?**, **?** and **?** among others. It is well known that if the tail of the i.i.d. random variables $X_1, \ldots, X_n$

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satisfies a stronger assumption than regularly varying with index $-\alpha$, known as second order regular variation, then

$$\sqrt{k}\left(h(k,n)-\frac{1}{\alpha}\right)\Longrightarrow N\left(0,\frac{1}{\alpha^2}\right).$$

While there are real life phenomena that do exhibit the presence of heavy tails, in lot of the cases there is a physical upper bound on the possible values. For example most internet service providers put an upper bound on the size of a file that can be transferred using an internet connection provided by them. Clearly the natural model for such phenomena is a truncated heavy-tailed distribution, a distribution which fits a heavy-tailed distribution till a certain point and then decays significantly faster. This can be made precise in the following way. Suppose that $H, H_1, \ldots$ are i.i.d. random variables so that $P(H > \cdot)$ is regularly varying with index $-\alpha$, $\alpha > 0$ and that $L, L_1, L_2, \ldots$ are i.i.d. random variables independent of $(H, H_1, H_2, \ldots)$. All these random variables are assumed to take values in the positive half line. We observe the sample $X_1, \ldots, X_n$ given by

$$X_j := H_j\mathbf{1}(H_j \le M_n) + (M_n + L_j)\mathbf{1}(H_j > M_n)\,, \tag{1.2}$$

where M_n, representing the truncating threshold, is a sequence of positive numbers going to infinity. Strictly speaking, the model is actually a triangular array $\{X_{nj} : 1 \le j \le n\}$. However, in practice we shall observe only one row of the triangular array, and hence we denote the sample by the usual notation $X_1, \ldots, X_n$. The random variable L can be thought of to have a much lighter tail, a tail decaying exponentially fast for example. However the results of this article are true under milder assumptions.

It was observed in Chakrabarty and Samorodnitsky (2009) that if the sequence M_n goes to infinity slow enough so that

$$\lim_{n\to\infty} nP(H > M_n) = \infty\,, \tag{1.3}$$

then a priori choosing a k so that the Hill estimator is consistent is a problem. In order to overcome that problem, the following sample based choice of k was suggested in that paper:

$$\hat{k}_n := \left[n\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}(X_j > \gamma X_{(1)})\right)^{\beta}\right], \tag{1.4}$$

where $\beta, \gamma \in (0,1)$ are user determined parameters. It has been shown in that article that this choice of $\hat{k}_n$ leads to a consistent estimator of α^{-1} when (1.3) is true, or when that limit is zero.

In this paper, we investigate the second order behavior of $h(\hat{k}_n, n)$ under the assumption (1.3) and some additional assumptions. We hope to address the case when the corresponding limit is zero in future. Knowing the second order behavior of an estimator, at least asymptotically, helps in constructing confidence intervals for the unknown parameter. While the problem is motivated by statistics, it is an interesting mathematical problem in itself. The complexity in analyzing the second order behavior of $h(\hat{k}_n, n)$ arises from the fact that now we are dealing with a random sum, and the number of summands is heavily dependent on the summands themselves. Also, a quick inspection will reveal that conditioning on the number of summands will completely destroy the i.i.d. nature of the sample, and thus make the analysis even more difficult.

In Section 2, it is shown that under some assumptions, the Hill estimator with $k = \hat{k}_n$ is asymptotically normal with mean $1/\alpha$. In Section 3, we connect the assumptions of Section 2 to the second order regular variation of the tail of H.

2. Asymptotic normality of the Hill estimator

Suppose that we have a one-dimensional non-negative sample $X_1, \ldots, X_n$ given by (1.2). We shall assume the following throughout this section.

Assumption A: There exists a sequence (ε_n) such that

$$\lim_{n\to\infty} P(H > M_n)^{-(1-\beta)}\varepsilon_n = 0\,, \tag{2.1}$$

$$\lim_{n\to\infty} nP(H > M_n)P(L > \varepsilon_n M_n) = 0\,, \tag{2.2}$$

$$\text{and } \lim_{n\to\infty} P(H > M_n)^{-(1-\beta)}\left\{\frac{l\left(\gamma M_n(1+\varepsilon_n)\right)}{l(\gamma M_n)} - 1\right\} = 0\,, \tag{2.3}$$

where $l(x) := x^\alpha P(H > x)$.

Assumption B: $\lim_{n\to\infty} nP(H > M_n) = \infty$.

Assumption C: $\lim_{n\to\infty} nP(H > M_n)^{2-\beta}(\log M_n)^2 = 0$.

Assumption D: For any sequence (v_n) satisfying

$$v_n \sim nP(H > \gamma M_n)^\beta\,, \tag{2.4}$$

it holds that

$$\lim_{n\to\infty}\sqrt{v_n}\left[\frac{n}{v_n}P\left(H > b(n/v_n)y^{-1/\alpha}\right) - y\right] = 0$$

uniformly on compact sets in $[0,\infty)$, where

$$b(y) := \inf\left\{x : \frac{1}{P(H > x)} \ge y\right\}\,. \tag{2.5}$$

Assumption E: For any sequence (v_n) satisfying (2.4), it is true that

$$\lim_{T\to\infty}\limsup_{n\to\infty}\sqrt{v_n}\int_T^\infty\left|\frac{n}{v_n}P\left(H > b(n/v_n)s\right) - s^{-\alpha}\right|\frac{ds}{s} = 0\,.$$

The main result of this section, Theorem 2.1, describes the second order behavior of $h(\hat{k}_n, n)$, where $h(\cdot,\cdot)$ and $\hat{k}_n$ are as defined in (1.1) and (1.4) respectively, under the assumptions A-E. Of course, these assumptions are hard to check in practice. However, in Section 3, we show that most of these can be verified if the tail of H is second order regularly varying and some additional conditions are satisfied. One could thus state the hypothesis of Theorem 2.1 in terms of the second order regular variation. The only reason why we decided not to do that is the following. The simplest example of a distribution with a regularly varying tail is a Pareto, which is known to not satisfy the second order regular variation as defined in **?**. Hence, if Theorem 2.1 is stated in terms of second order regular variation, it will not entail simple examples of regularly varying distributions like! Pareto, which clearly satisfy the assumptions A, D and E.

Theorem 2.1. *Under assumptions A,B,C,D and E,*

$$\sqrt{\hat{k}_n}\left\{h(\hat{k}_n, n) - \frac{1}{\alpha}\right\} \Longrightarrow N\left(0, \frac{1}{\alpha^2}\right). \tag{2.6}$$

The following is a brief outline of how we plan to prove this. Define

$$\begin{aligned} U_n &:= \sum_{j=1}^{n} \mathbf{1}(X_j > \gamma M_n)\,, \\ V_n &:= \sum_{j=1}^{n} \mathbf{1}(X_j > \gamma X_{(1)})\,, \\ \tilde{k}_n &:= \left[n^{1-\beta} U_n^{\beta}\right]. \end{aligned}$$

Note that

$$\hat{k}_n := \left[n^{1-\beta} V_n^{\beta}\right].$$

Since we are dealing with a random sum, a natural way of proceeding is conditioning on the number of summands. However, as commented earlier, conditioning on V_n or $\hat{k}_n$ destroys the i.i.d. nature of the sample. Hence, we condition on $U_n = u_n$, where (u_n) is any sequence of integers satisfying $u_n \sim nP(H > \gamma M_n)$. Lemma 2.1 is a general result, which allows us to claim weak convergence of the unconditional distribution based on that of the conditional distribution. Clearly, by conditioning on U_n, $h(\tilde{k}_n, n)$ becomes the Hill statistic with a deterministic k applied to a triangular array. The second order behavior of that is studied in Lemma 2.3. In view of Lemma 2.1, this translates to second order behavior of (the unconditional distribution of) $h(\tilde{k}_n, n)$. In order to argue the claim of Theorem 2.1, all we need is showing that $h(\tilde{k}_n, n)$ and $h(\hat{k}_n, n)$ are not very far apart, and that is done in Lemma 2.4. For Lemma 2.3 and Lemma 2.4, we need that the tail empirical process, after suitable centering and scaling, converge to a Brownian Motion. This has been showed in Lemma 2.2.

Lemma 2.1. *Suppose that $(B_n : n \geq 1)$ is a sequence of discrete random variables satisfying*

$$\frac{B_n}{b_n} \xrightarrow{P} 1\,,$$

for some deterministic sequence (b_n). Assume that $(A_n : n \geq 1)$ is a family of random variables such that whenever $\hat{b}_n$ is any deterministic sequence satisfying $\hat{b}_n \sim b_n$ as $n \longrightarrow \infty$ and $P(B_n = \hat{b}_n) > 0$, it follows that

$$P(A_n \leq \cdot | B_n = \hat{b}_n) \Longrightarrow F(\cdot)\,, \tag{2.7}$$

for some c.d.f. F. Then $A_n \Longrightarrow F$.

Proof. It suffices to show that every subsequence of (A_n) has a further subsequence that converges weakly to F. Since every sequence that converges in probability has a subsequence that converges almost surely, we can assume without loss of generality that

$$\frac{B_n}{b_n} \longrightarrow 1 \text{ a.s.}\,. \tag{2.8}$$

Fix a continuity point x of F and define a function $f_n : \mathbb{R} \longrightarrow [0,1]$ by

$$f_n(u) = \begin{cases} \frac{P(A_n \le x, B_n = u)}{P(B_n = u)}, & \text{if } P(B_n = u) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, for all $n \ge 1$,

$$P(A_n \le x) = Ef_n(B_n)\,.$$

By (2.7) and (2.8), it follows that

$$f_n(B_n) \longrightarrow F(x) \text{ a.s.}\,.$$

By the bounded convergence theorem, it follows that

$$\lim_{n\to\infty} Ef_n(B_n) = F(x)\,,$$

and this completes the proof. □

Throughout this section, assumptions A, B, C, D and E will be in force.

Lemma 2.2. *Suppose that (u_n) is a sequence of integers satisfying*

$$u_n \sim nP(H > \gamma M_n)\,, \tag{2.9}$$

and let

$$v_n := [n^{1-\beta} u_n^{\beta}] - u_n\,, \tag{2.10}$$

$$\tilde{M}_n := \gamma M_n\,. \tag{2.11}$$

Let for $n \ge 1$, $Y_{n,1}, \ldots, Y_{n,n}$ be i.i.d. with c.d.f. F_n, defined as

$$F_n(x) := P(H \le x | H \le \tilde{M}_n)\,.$$

Then,

$$\sqrt{v_n}\left(\frac{1}{v_n}\sum_{i=1}^{n-u_n} \delta_{Y_{n-u_n,i}/b((n-u_n)/v_n)}(y^{-1/\alpha}, \infty] - y\right) \Longrightarrow W(y) \tag{2.12}$$

in $D[0,\infty)$, where $D[0,\infty)$ is endowed with the topology of uniform convergence on compact sets and W is the standard Brownian Motion on $[0,\infty)$.

Proof. For simplicity sake, denote $w_n := n - u_n$. It is easy to see by assumptions B and C that

$$1 \ll w_n P(H > \tilde{M}_n) \ll \sqrt{v_n} \ll \sqrt{w_n}\,. \tag{2.13}$$

Let $(\Gamma_i : i \geq 1)$ be the arrivals of a unit rate Poisson Process. Define

$$\phi_n(s) := \frac{\Gamma_{w_n+1}}{v_n}\bar{F}_n(s^{-1/\alpha} b(w_n/v_n))\,,$$

where $\bar{G} := 1 - G$ for any function G. By the discussion on page 24 in **?**, it follows that

$$\lim_{n\to\infty} \frac{w_n}{v_n} P(H > b(w_n/v_n)) = 1\,. \tag{2.14}$$

It follows by (2.13) that

$$\lim_{n\to\infty} \frac{w_n}{v_n} P(H > \tilde{M}_n) = 0\,.$$

This in conjunction with (2.14) implies that

$$b(w_n/v_n) = o(\tilde{M}_n)\,.$$

It is easy to see that v_n satisfies (2.4). Hence, for n large enough,

$$\begin{aligned} &\frac{w_n}{v_n}\bar{F}_n(s^{-1/\alpha} b(w_n/v_n)) - s \\ = \; & \frac{1}{P(H \leq \tilde{M}_n)}\Bigg[\frac{w_n}{v_n} P\left(H > s^{-1/\alpha} b(w_n/v_n)\right) - \frac{w_n}{v_n} P(H > \tilde{M}_n) \\ & -s + sP(H > \tilde{M}_n)\Bigg]\,, \end{aligned}$$

and hence in view of Assumption D and (2.13), it follows that for $0 < T < \infty$,

$$\lim_{n\to\infty} \sqrt{v_n} \sup_{0\leq s\leq T} \left|\frac{w_n}{v_n}\bar{F}_n(s^{-1/\alpha} b(w_n/v_n)) - s\right| = 0\,. \tag{2.15}$$

Also note that,

$$\begin{aligned} & \sup_{0\leq s\leq T}\left|\phi_n(s) - \frac{w_n}{v_n}\bar{F}_n(s^{-1/\alpha} b(w_n/v_n))\right| \\ = \; & \left|\frac{\Gamma_{w_n+1}}{w_n} - 1\right| \frac{w_n}{v_n}\bar{F}_n(T^{-1/\alpha} b(w_n/v_n)) \\ = \; & O_p(w_n^{-1/2})O(1) \\ = \; & o_p(v_n^{-1/2})\,. \end{aligned}$$

This in conjunction with (2.15) shows that

$$\sqrt{v_n}\,(\phi_n(s) - s) \xrightarrow{P} 0 \tag{2.16}$$

in $D[0,\infty)$. Recall that since $1 \ll v_n \ll w_n$, in $D[0,\infty)$,

$$\sqrt{v_n}\left(\frac{1}{v_n}\sum_{i=1}^{w_n} \mathbf{1}\left(\Gamma_i \leq v_n s\right) - s\right) \Longrightarrow W(s)\,;$$

see (9.7), page 294 in **?**. Hence, it follows by the continuous mapping theorem and Slutsky's theorem that

$$\sqrt{v_n}\left(\frac{1}{v_n}\sum_{i=1}^{w_n}\mathbf{1}\left(\Gamma_i \le v_n\phi_n(s)\right) - \phi_n(s)\right) \Longrightarrow W(s) \tag{2.17}$$

in $D[0,\infty)$. By similar arguments as those in the proof of Theorem 9.1 in **?**, it follows that

$$\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(y^{-1/\alpha},\infty] \stackrel{d}{=} \sum_{i=1}^{w_n}\mathbf{1}\left(\Gamma_i \le v_n\phi_n(s)\right) .$$

This along with (2.16) and (2.17) shows (2.12). □

Lemma 2.3. *Let* (u_n) *be a sequence of integers satisfying* (2.9) *and let* (v_n) *and* $(\tilde{M}_n)$ *be as defined in* (2.10) *and* (2.11) *respectively. Then,*

$$\sqrt{v_n}\left(\frac{1}{v_n}\sum_{i=1}^{v_n}\log\frac{Y_{(n-u_n,i)}}{Y_{(n-u_n,v_n)}} - \frac{1}{\alpha}\right) \Longrightarrow N\left(0,\frac{1}{\alpha^2}\right) ,$$

where $Y_{(n,1)} \ge \ldots \ge Y_{(n,n)}$ *are the order statistics of* $Y_{n,1},\ldots,Y_{n,n}$*, and the latter is as defined in Lemma 2.2.*

Proof. Once again, let us denote $w_n := n - u_n$. An application of Vervaat's lemma (Proposition 3.3 in **?**) to (2.12) shows that

$$\sqrt{v_n}\left[\left\{\frac{Y_{(w_n,v_n)}}{b(w_n/v_n)}\right\}^{-\alpha} - 1\right] \Longrightarrow -W(1) \tag{2.18}$$

jointly with (2.12). This in particular, shows that

$$\left(\sqrt{v_n}\left\{\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(x,\infty] - x^{-\alpha}\right\}, \frac{Y_{(w_n,v_n)}}{b(w_n/v_n)}\right)$$
$$\Longrightarrow \left(W(x^{-\alpha}),1\right),$$

in $D(0,\infty]\times\mathbb{R}$, jointly with (2.18), where $D(0,\infty]$ is also endowed with the topology of uniform convergence on compact sets. Using the continuous mapping theorem, it follows that

$$\sqrt{v_n}\left\{\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/Y_{(w_n,v_n)}}(x,\infty] - x^{-\alpha}\frac{Y_{(w_n,v_n)}^{-\alpha}}{b(w_n/v_n)^{-\alpha}}\right\}$$
$$\Longrightarrow W(x^{-\alpha}) , \tag{2.19}$$

in $D(0,\infty]$, jointly with (2.18). As in the proof of Proposition 9.1 in **?**, we shall apply the map ψ from $D(0,\infty]$ to $\mathbb{R}$, defined by

$$\psi(f) := \int_1^\infty f(s)\frac{ds}{s} ,$$

to conclude that

$$\sqrt{v_n}\left\{\frac{1}{v_n}\sum_{i=1}^{v_n}\log\frac{Y_{(w_n,i)}}{Y_{(w_n,v_n)}}-\frac{1}{\alpha}\frac{Y_{(w_n,v_n)}^{-\alpha}}{b(w_n/v_n)^{-\alpha}}\right\}\Longrightarrow\int_1^\infty W(x^{-\alpha})\frac{dx}{x}\,, \tag{2.20}$$

jointly with (2.18). This implies that

$$\sqrt{v_n}\left\{\frac{1}{v_n}\sum_{i=1}^{v_n}\log\frac{Y_{(n,i)}}{Y_{(n,v_n)}}-\frac{1}{\alpha}\right\}\Longrightarrow\int_1^\infty W(x^{-\alpha})\frac{dx}{x}-\frac{1}{\alpha}W(1)$$

as desired. Thus, it suffices to show (2.20).

To that end, note that for $1<T<\infty$, the map ψ_T, defined by

$$\psi_T(f):=\int_1^T f(s)\frac{ds}{s}$$

is continuous and has compact support. Also, as $T\longrightarrow\infty$,

$$\psi_T(W(s^{-\alpha}))\Longrightarrow\psi(W(s^{-\alpha}))\,.$$

Some calculations will show that ψ applied to the left hand side of (2.19) gives the left hand side of (2.20). Thus, all that needs to be done is justifying the application of ψ to (2.19), and for that, it suffices to check that for all $\epsilon>0$,

$$\lim_{T\to\infty}\limsup_{n\to\infty}P\left[\sqrt{v_n}\int_T^\infty\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/Y_{(w_n,v_n)}}(x,\infty]\right.\right.$$
$$\left.\left.-x^{-\alpha}\frac{Y_{(w_n,v_n)}^{-\alpha}}{b(w_n/v_n)^{-\alpha}}\right|\frac{dx}{x}>\epsilon\right]=0\,.$$

Note that on the set $\{Y_{(w_n,v_n)}/b(w_n/v_n)>1/2\}$,

$$\int_T^\infty\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/Y_{(w_n,v_n)}}(x,\infty]-x^{-\alpha}\frac{Y_{(w_n,v_n)}^{-\alpha}}{b(w_n/v_n)^{-\alpha}}\right|\frac{dx}{x}$$
$$=\int_{TY_{(w_n,v_n)}/b(w_n/v_n)}^\infty\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]-u^{-\alpha}\right|\frac{du}{u}$$
$$\le\int_{T/2}^\infty\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]-u^{-\alpha}\right|\frac{du}{u}\,.$$

Since $P[Y_{(w_n,v_n)}/b(w_n/v_n)\le 1/2]$ goes to zero, it suffices to show that

$$\lim_{T\to\infty}\limsup_{n\to\infty}P\left[\sqrt{v_n}\int_{T/2}^\infty\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]\right.\right.$$
$$\left.\left.-u^{-\alpha}\right|\frac{du}{u}>\epsilon\right]=0\,. \tag{2.21}$$

Clearly,

$$
\begin{aligned}
&\int_{T/2}^{\infty}\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]-u^{-\alpha}\right|\frac{du}{u}\\
\le\ &\int_{T/2}^{\infty}\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]-\frac{w_n}{v_n}\bar F_n\left(ub(w_n/v_n)\right)\right|\frac{du}{u}\\
&+\frac{w_n}{v_n}\int_{T/2}^{\infty}\left|\bar F_n\left(ub(w_n/v_n)\right)-P\left(H>ub(w_n/v_n)\right)\right|\frac{du}{u}\\
&+\int_{T/2}^{\infty}\left|\frac{w_n}{v_n}P\left(H>ub(w_n/v_n)\right)-u^{-\alpha}\right|\frac{du}{u}\\
=\ &\int_{T/2}^{\infty}\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]-\frac{w_n}{v_n}\bar F_n\left(ub(w_n/v_n)\right)\right|\frac{du}{u}\\
&+\frac{w_n}{v_n}\int_{T/2}^{\tilde M_n/b(w_n/v_n)}\left|\bar F_n\left(ub(w_n/v_n)\right)-P\left(H>ub(w_n/v_n)\right)\right|\frac{du}{u}\\
&+\frac{w_n}{v_n}\int_{\tilde M_n}^{\infty}P(H>u)\frac{du}{u}\\
&+\int_{T/2}^{\infty}\left|\frac{w_n}{v_n}P\left(H>ub(w_n/v_n)\right)-u^{-\alpha}\right|\frac{du}{u}\\
=:\ &I_1+I_2+I_3+I_4\,.
\end{aligned}
$$

Since v_n is defined by (2.10), (2.4) holds. By Assumption E, it follows that

$$\lim_{T\to\infty}\limsup_{n\to\infty}\sqrt{v_n}I_4=0\,.$$

Karamata's theorem (Theorem VIII.9.1, page 281 in **?**) implies that

$$I_3=O\left(\frac{w_n}{v_n}P(H>\tilde M_n)\right)=o\left(v_n^{-1/2}\right)\,,$$

the second equality following from (2.13). For I_2, note that

$$
\begin{aligned}
&\bar F_n\left(ub(w_n/v_n)\right)-P\left(H>ub(w_n/v_n)\right)\\
=\ &-\frac{P(H>\tilde M_n)P\left(H\le ub(w_n/v_n)\right)}{P(H\le\tilde M_n)}\,.
\end{aligned}
$$

Also, it is easy to see from assumption C that

$$\lim_{n\to\infty}\frac{w_nP(H>\tilde M_n)}{\sqrt{v_n}}\log\left\{\frac{\tilde M_n}{b(w_n/v_n)}\right\}=0\,. \tag{2.22}$$

Thus,

$$I_2=O\left(\frac{w_n}{v_n}P(H>\tilde M_n)\log\frac{\tilde M_n}{b(w_n/v_n)}\right)=o\left(v_n^{-1/2}\right)\,,$$

the second equality following from (2.22).

Thus, all that remains is showing

$$\lim_{T\to\infty}\limsup_{n\to\infty} P[\sqrt{v_n} I_1 > \epsilon] = 0\,. \tag{2.23}$$

Notice that

$$E\left[\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]\right] = \frac{w_n}{v_n}\bar{F}_n\left(ub(w_n/v_n)\right)\,.$$

Letting C to be a finite positive constant independent of n, whose value may change from line to line,

$$\begin{aligned}
& P[\sqrt{v_n} I_1 > \epsilon] \\
\le\ & \frac{\sqrt{v_n}}{\epsilon}E(I_1) \\
=\ & C\sqrt{v_n}\int_{T/2}^{\infty} E\left|\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty] - \frac{w_n}{v_n}\bar{F}_n\left(ub(w_n/v_n)\right)\right|\frac{du}{u} \\
\le\ & C\sqrt{v_n}\int_{T/2}^{\infty}\mathrm{Var}\left[\frac{1}{v_n}\sum_{i=1}^{w_n}\delta_{Y_{w_n,i}/b(w_n/v_n)}(u,\infty]\right]^{1/2}\frac{du}{u} \\
\le\ & C\frac{\sqrt{w_n}}{\sqrt{v_n}}\int_{T/2}^{\infty}\bar{F}_n\left(ub(w_n/v_n)\right)^{1/2}\frac{du}{u} \\
\le\ & C\int_{T/2}^{\infty}\frac{\sqrt{w_n}}{\sqrt{v_n}}P\left(H > ub(w_n/v_n)\right)^{1/2}\frac{du}{u}\,.
\end{aligned}$$

By (2.14), the integrand clearly converges to $u^{-\alpha/2}$ as $n \longrightarrow \infty$. By (2.5), the integrand is bounded above by

$$\left[\frac{P(H > ub(w_n/v_n))}{P(H > b(w_n/v_n))}\right]^{1/2}\,,$$

which by the Potter bounds (Proposition 2.6 in **?**) is bounded above by $2u^{-\alpha/3}$ for n large enough. An appeal to the dominated convergence theorem shows (2.23) and thus completes the proof. □

Lemma 2.4. *As $n \longrightarrow \infty$,*

$$\sqrt{\tilde{k}_n}\left\{h(\tilde{k}_n, n) - h(\hat{k}_n, n)\right\} \xrightarrow{P} 0\,. \tag{2.24}$$

Proof. We start with showing that

$$\sqrt{\hat{k}_n}\left[\frac{\hat{k}_n}{\tilde{k}_n} - 1\right] \xrightarrow{P} 0\,. \tag{2.25}$$

In the proof of Theorem 3.2 in Chakrabarty and Samorodnitsky (2009), it has been shown that under Assumption B,

$$\frac{U_n}{nP(H > \gamma M_n)} \xrightarrow{P} 1\,, \tag{2.26}$$

$$\frac{V_n}{nP(H > \gamma M_n)} \xrightarrow{P} 1\,, \tag{2.27}$$

$$\text{and } \frac{\hat{k}_n}{nP(H > \gamma M_n)^\beta} \xrightarrow{P} 1\,. \tag{2.28}$$

In view of (2.28), it suffices to show that

$$n^{1/2}P(H > M_n)^{\beta/2}\left[\frac{\hat{k}_n}{\tilde{k}_n} - 1\right] \xrightarrow{P} 0\,.$$

Note that,

$$\frac{n^{1-\beta}V_n^\beta}{n^{1-\beta}U_n^\beta + 1} \le \frac{\hat{k}_n}{\tilde{k}_n} \le \frac{n^{1-\beta}V_n^\beta + 1}{n^{1-\beta}U_n^\beta}\,,$$

$$\frac{n^{1-\beta}V_n^\beta}{n^{1-\beta}U_n^\beta + 1} \le \left(\frac{V_n}{U_n}\right)^\beta \le \frac{n^{1-\beta}V_n^\beta + 1}{n^{1-\beta}U_n^\beta}\,,$$

and

$$\begin{aligned}
\frac{n^{1-\beta}V_n^\beta + 1}{n^{1-\beta}U_n^\beta} - \frac{n^{1-\beta}V_n^\beta}{n^{1-\beta}U_n^\beta + 1} &= \frac{n^{1-\beta}V_n^\beta + n^{1-\beta}U_n^\beta + 1}{n^{1-\beta}U_n^\beta(n^{1-\beta}U_n^\beta + 1)} \\
&= O_p\left(n^{-1}P(H > M_n)^{-\beta}\right) \\
&= o_p\left(n^{-1/2}P(H > M_n)^{-\beta/2}\right)\,,
\end{aligned}$$

the equality in the second line following from (2.26) and (2.27), and that in the third line following from Assumption B. Thus, it suffices to show that

$$n^{1/2}P(H > M_n)^{\beta/2}\left[\left(\frac{V_n}{U_n}\right)^\beta - 1\right] \xrightarrow{P} 0\,.$$

By the mean value theorem, it follows that as $x \longrightarrow 1$,

$$x^\beta - 1 = O(|x - 1|)\,.$$

Hence, in view of the fact that V_n/U_n converges to 1 in probability, it suffices to show that

$$n^{1/2}P(H > M_n)^{\beta/2}\left(\frac{V_n}{U_n} - 1\right) \xrightarrow{P} 0\,.$$

Using (2.26) once again, all that needs to be shown is

$$V_n - U_n = o_p\left(n^{-1/2}P(H > M_n)^{-(1-\beta/2)}\right)\,.$$

Note that on the set $\{M_n \le X_{(1)} \le M_n(1+\varepsilon_n)\}$, where ε_n is chosen to satisfy Assumption A,

$$0 \le U_n - V_n \le \sum_{j=1}^{n} \mathbf{1}\left(\gamma M_n < X_j \le \gamma M_n(1+\varepsilon_n)\right) =: T_n\,.$$

Thus, it suffices to show that

$$\lim_{n\to\infty} P(X_{(1)} \le M_n(1+\varepsilon_n)) = 1\,, \tag{2.29}$$

$$\lim_{n\to\infty} P(X_{(1)} \ge M_n) = 1\,, \tag{2.30}$$

$$\text{and } T_n = o_p\left(n^{-1/2} P(H > M_n)^{-(1-\beta/2)}\right)\,. \tag{2.31}$$

For (2.29), note that as $n \longrightarrow \infty$,

$$P(X_{(1)} \le M_n(1+\varepsilon_n)) \;=\; (1 - P(H > M_n)P(L > \varepsilon_n M_n))^n \longrightarrow 1\,,$$

the convergence following from (2.2) in Assumption A. This shows (2.29). For (2.30), observe that

$$P(X_{(1)} < M_n) \le (1 - P(H > M_n))^n\,.$$

By Assumption B, the right hand side converges to zero, and hence (2.30) holds. To show (2.31), note that

$$\mathrm{Var}(T_n) \le E(T_n) = np_n\,,$$

where

$$p_n := P(\gamma M_n < X_1 \le \gamma(1+\varepsilon_n)M_n)\,.$$

In view of Assumption C, for (2.31), it suffices to show that

$$p_n = o(P(H > M_n)^{2-\beta})\,. \tag{2.32}$$

For n large enough so that $\gamma(1+\varepsilon_n) < 1$,

$$\begin{aligned} p_n &= P(H > \gamma M_n) - \gamma^{-\alpha} M_n^{-\alpha}(1+\varepsilon_n)^{-\alpha} l\left(\gamma M_n(1+\varepsilon_n)\right) \\ &= \gamma^{-\alpha} M_n^{-\alpha} l\left(\gamma M_n(1+\varepsilon_n)\right)\left\{1 - (1+\varepsilon_n)^{-\alpha}\right\} \\ &\quad + P(H > \gamma M_n)\left\{1 - \frac{l\left(\gamma M_n(1+\varepsilon_n)\right)}{l(\gamma M_n)}\right\}\,. \end{aligned}$$

The first term on the right hand side is clearly $O(\varepsilon_n P(H > M_n))$, which by (2.1), is $o\left(P(H > M_n)^{2-\beta}\right)$. By (2.3), it follows that the second term is also $o\left(P(H > M_n)^{2-\beta}\right)$. This shows (2.32), and thus completes the proof of (2.25).

Next, we show that for all $\eta \in \mathbb{R}$, as $n \longrightarrow \infty$,

$$\sqrt{\tilde{k}_n} \log \frac{X_{(n,[\tilde{k}_n + \eta \tilde{k}_n^{1/2}])}}{X_{(n,\tilde{k}_n)}} \xrightarrow{P} -\frac{\eta}{\alpha}\,. \tag{2.33}$$

Let (u_n) be a sequence of positive integers satisfying (2.9) For n large enough so that $1 \le u_n < [n^{1-\beta}u_n^\beta] \le n$ and $1 \le u_n < [n^{1-\beta}u_n^\beta] + \eta[n^{1-\beta}u_n^\beta]^{1/2} \le n$, the conditional distribution of $\left(X_{(\tilde{k}_n)}, X_{([\tilde{k}_n+\eta\tilde{k}_n^{1/2}])}\right)$ given that $U_n = u_n$ is same as the (unconditional) distribution of

$$\left(Y_{(n-u_n,[n^{1-\beta}u_n^\beta]-u_n)}, Y_{(n-u_n,[n^{1-\beta}u_n^\beta]+\eta[n^{1-\beta}u_n^\beta]^{1/2}-u_n)}\right),$$

where $\{Y_{(n,j)} : 1 \le j \le n\}$ is as defined in Lemma 2.2, with $\tilde{M}_n$ as in (2.11). Define v_n as in (2.10) By Lemma 2.2, it follows that

$$\sqrt{v_n}\left(\frac{1}{v_n}\sum_{i=1}^n \delta_{Y_{n-u_n,i}/b((n-u_n)/v_n)}(y^{-1/\alpha},\infty] - y\right) \Longrightarrow W(y)$$

in $D[0,\infty)$. Using Vervaat's lemma, it follows that

$$\sqrt{v_n}\left[\left(\frac{Y_{(n-u_n,[v_n x])}}{b((n-u_n)/v_n)}\right)^{-\alpha} - x\right] \Longrightarrow -W(x) \tag{2.34}$$

in $D[0,\infty)$. From here, we conclude that

$$\left(\sqrt{v_n}\left[\left(\frac{Y_{(n-u_n,[v_n s_n])}}{b((n-u_n)/v_n)}\right)^{-\alpha} - s_n\right], \sqrt{v_n}\left[\left(\frac{Y_{(n-u_n,v_n)}}{b((n-u_n)/v_n)}\right)^{-\alpha} - 1\right]\right)$$

$$\Longrightarrow (-W(1), -W(1)),$$

where $s_n := 1 + \eta v_n^{-1}[n^{1-\beta}u_n^\beta]^{1/2}$. Since the limit process is $C[0,\infty) \times C[0,\infty)$ valued, this can be done using Skorohod's Theorem (Theorem 2.2.2 in **?**). Using the Delta method with $x \mapsto -\frac{1}{\alpha}\log x$, it follows that

$$\left(\sqrt{v_n}\left\{\log\frac{Y_{(n-u_n,[v_n s_n])}}{b((n-u_n)/v_n)} + \frac{1}{\alpha}\log s_n\right\}, \sqrt{v_n}\log\frac{Y_{(n-u_n,v_n)}}{b((n-u_n)/v_n)}\right)$$

$$\Longrightarrow \left(\frac{1}{\alpha}W(1), \frac{1}{\alpha}W(1)\right).$$

Since,

$$\lim_{n\to\infty}\sqrt{v_n}\log s_n = \eta,$$

it follows that

$$\sqrt{v_n}\log\frac{Y_{(n-u_n,[n^{1-\beta}u_n^\beta]-u_n)}}{Y_{(n-u_n,[n^{1-\beta}u_n^\beta]+\eta[n^{1-\beta}u_n^\beta]^{1/2}-u_n)}} \xrightarrow{P} -\frac{\eta}{\alpha}.$$

What we have shown is that whenever (u_n) is a sequence satisfying (2.9), the conditional distribution of the left hand side of (2.33) given $U_n = u_n$ converges weakly to $-\eta/\alpha$. By an appeal to Lemma 2.1, this shows (2.33).

Coming to the proof of (2.24), note that

$$\begin{aligned}
&\sqrt{\tilde{k}_n}\left[h(\hat{k}_n, n) - h(\tilde{k}_n, n)\right] \\
= \; & \frac{1}{\sqrt{\tilde{k}_n}}\left[\sum_{i=1}^{\hat{k}_n}\log\frac{X_{(i)}}{X_{(\tilde{k}_n)}} - \sum_{i=1}^{\tilde{k}_n}\log\frac{X_{(i)}}{X_{(\tilde{k}_n)}}\right] + \frac{\hat{k}_n}{\sqrt{\tilde{k}_n}}\log\frac{X_{(\tilde{k}_n)}}{X_{(\hat{k}_n)}} \\
& + \sqrt{\tilde{k}_n}\left(\frac{1}{\hat{k}_n} - \frac{1}{\tilde{k}_n}\right)\sum_{i=1}^{\hat{k}_n}\log\frac{X_{(i)}}{X_{(\hat{k}_n)}} \\
=: \; & A + B + C\,.
\end{aligned}$$

Clearly,

$$C = \sqrt{\tilde{k}_n}\left(1 - \frac{\hat{k}_n}{\tilde{k}_n}\right)h(\hat{k}_n, n) \xrightarrow{P} 0\,,$$

the convergence in probability following from (2.25) and the fact that

$$h(\hat{k}_n, n) \xrightarrow{P} 1/\alpha\,,$$

which has been shown in **?**. For showing that $B \xrightarrow{P} 0$, fix $\epsilon > 0$ and let $\eta := \epsilon\alpha/6$. Note that

$$\begin{aligned}
&P(|B| > \epsilon) \\
\le P\left[\frac{\hat{k}_n}{\tilde{k}_n} > 2\right] + P\left[\sqrt{\tilde{k}_n}\left|\frac{\hat{k}_n}{\tilde{k}_n} - 1\right| > \eta\right] + P\left[\sqrt{\tilde{k}_n}\log\frac{X_{(\tilde{k}_n - \eta\tilde{k}_n^{1/2})}}{X_{(\tilde{k}_n + \eta\tilde{k}_n^{1/2})}} > 3\frac{\eta}{\alpha}\right].
\end{aligned}$$

By (2.25) and (2.33), it follows that $B \xrightarrow{P} 0$. Since for $0 < \epsilon < 1$,

$$P(|A| > \epsilon) \le P\left[\sqrt{\tilde{k}_n}\left|\frac{\hat{k}_n}{\tilde{k}_n} - 1\right| > \epsilon\right] + P\left[\log\frac{X_{(\tilde{k}_n - \tilde{k}_n^{1/2})}}{X_{(\tilde{k}_n + \tilde{k}_n^{1/2})}} > 1\right],$$

it is immediate that $A \xrightarrow{P} 0$. This completes the proof. □

Proof of Theorem 2.1. In view of Lemma 2.4, it suffices to show that

$$\sqrt{\tilde{k}_n}\left(h(\tilde{k}_n, n) - \frac{1}{\alpha}\right) \Longrightarrow N\left(0, \frac{1}{\alpha^2}\right). \tag{2.35}$$

Define

$$\begin{aligned}
S_1 &:= \sum_{i=1}^{U_n}\log\frac{X_{(i)}}{X_{(\tilde{k}_n)}} \\
S_2 &:= \sum_{i=U_n+1}^{\tilde{k}_n}\log\frac{X_{(i)}}{X_{(\tilde{k}_n)}}
\end{aligned}$$

and note that on the set $\{U_n \le \tilde{k}_n\}$,

$$h(\tilde{k}_n, n) = \frac{1}{\tilde{k}_n}(S_1 + S_2)\,.$$

Let u_n be a sequence of integers satisfying (2.9) and define v_n and $\tilde{M}_n$ as in (2.10) and (2.11). For n large enough, note that

$$[S_2|U_n = u_n] \stackrel{d}{=} \sum_{i=1}^{v_n} \log \frac{Y_{(n-u_n,i)}}{Y_{(n-u_n,v_n)}} =: \tilde{S}_2 \,,$$

where $\{Y_{(n,j)} : 1 \le j \le n\}$ is as defined in the statement of Lemma 2.3. By Lemma 2.3, it follows that

$$\sqrt{v_n}\left(\frac{1}{v_n}\tilde{S}_2 - \frac{1}{\alpha}\right) \Longrightarrow N\left(0, \frac{1}{\alpha^2}\right) .$$

This along with the fact that

$$\sqrt{v_n}\tilde{S}_2\left(\frac{1}{[n^{1-\beta}u_n^\beta]} - \frac{1}{v_n}\right) = -\frac{\tilde{S}_2}{[n^{1-\beta}u_n^\beta]}\frac{u_n}{\sqrt{v_n}} = O_p(1)o(1)\,,$$

shows that

$$\left[\sqrt{\tilde{k}_n}\left(\frac{1}{\tilde{k}_n}S_2 - \frac{1}{\alpha}\right)\middle| U_n = u_n\right] \Longrightarrow N\left(0, \frac{1}{\alpha^2}\right) .$$

Since this is true for all sequence of integers (u_n) satisfying (2.9), by Lemma 2.1 it follows that

$$\sqrt{\tilde{k}_n}\left(\frac{1}{\tilde{k}_n}S_2 - \frac{1}{\alpha}\right) \Longrightarrow N\left(0, \frac{1}{\alpha^2}\right) .$$

On the set $\{1 \le X_{(1)} \le 2M_n\}$,

$$\begin{aligned} \frac{S_1}{\sqrt{\tilde{k}_n}} &\le \frac{U_n \log(2M_n)}{\sqrt{\tilde{k}_n}} \\ &= O_p\left(n^{1/2}P(H > M_n)^{1-\beta/2}\log M_n\right) \\ &= o_p(1)\,. \end{aligned}$$

Since the probability of that set converges to one, it follows that

$$\frac{S_1}{\sqrt{\tilde{k}_n}} \xrightarrow{P} 0\,.$$

This completes the proof. □

3. Second order regular variation

In this section, we show that if the tail of H is second order regularly varying, and L is sufficiently light-tailed, then the hypotheses of Theorem 2.1 hold. By the tail being second order regularly varying, we mean that there is a function $A : (0, \infty) \longrightarrow (0, \infty)$ which is regularly varying with index $\rho\alpha$ where $\rho < 0$, such that

$$\lim_{t\to\infty} \frac{\frac{P(H>tx)}{P(H>t)} - x^{-\alpha}}{A(t)} = x^{-\alpha}\frac{x^{\rho\alpha} - 1}{\rho/\alpha} \tag{3.1}$$

for all $x > 0$; see (2.3.24) in **?**.

Theorem 3.1. *Suppose that*

$$\max(1 - 1/\alpha, 0) < \beta < 1\,, \tag{3.2}$$

all moments of L are finite, M_n satisfies assumptions B and C, and the tail of H is second order regularly varying so that the second order parameter ρ satisfies

$$\rho < -\frac{1-\beta}{\beta}\,.$$

Then, (2.6) *holds.*

Proof. In view of Theorem 2.1, it suffices to check that assumptions A, D and E hold. By Theorem 2.3.9 in **?**, it follows that given $\epsilon, \delta > 0$, there exist $t_0 > 1$ such that whenever $t, tx \geq t_0$,

$$\left| \frac{\frac{P(H>tx)}{P(H>t)} - x^{-\alpha}}{A(t)} - x^{-\alpha}\frac{x^{\rho\alpha} - 1}{\rho/\alpha} \right| \leq \epsilon x^{-\alpha+\rho\alpha}\max(x^{\delta}, x^{-\delta})\,. \tag{3.3}$$

Note that (3.3) holds with a possibly different $A(t)$ from that in (3.1). However, this A is also regularly varying with index $\rho\alpha$. For the rest of the proof, by $A(\cdot)$, we shall mean the one for which (3.3) holds.

We start with showing that

$$\sqrt{v_n} = o\left(A(b(n/v_n))^{-1}\right)\,, \tag{3.4}$$

whenever v_n is a sequence satisfying (2.4). Let

$$\eta := -\rho\beta - (1-\beta)\,.$$

The upper bound on ρ implies $\eta > 0$. Note that $A(b(\cdot))$ varies regularly with index ρ and $n/v_n \sim P(H > \gamma M_n)^{-\beta}$. Thus, there is a slowly varying function $\bar{l}$ so that

$$\begin{aligned} A\left(b(n/v_n)\right)^{-1} &\sim \bar{l}(M_n)P(H > M_n)^{\rho\beta} \\ &\gg P(H > M_n)^{\eta+\rho\beta} \\ &= \frac{n^{1/2}P(H > M_n)^{\beta/2}}{n^{1/2}P(H > M_n)^{1-\beta/2}} \\ &\gg n^{1/2}P(H > M_n)^{\beta/2} \\ &\sim \gamma^{\alpha\beta/2}\sqrt{v_n}\,, \end{aligned}$$

the inequality in the second last line following from Assumption C. This shows (3.4).

Now, we show that assumptions D and E hold. Let

$$\varepsilon_n := A(b(n/v_n)) \wedge (1/2)\,.$$

Clearly $1 > \varepsilon_n > 0$ for all n. Recall from (2.5) that $z < b(y)$ if and only if $P(H > z)^{-1} < y$. Thus,

$$\frac{1}{P\left(H > (1-\varepsilon_n)b(n/v_n)\right)} < \frac{n}{v_n} \le \frac{1}{P\left(H > b(n/v_n)\right)}\,.$$

Let $\delta > 0$ be such that $\rho\alpha + \delta < 0$. Let t_0 be such that whenever $t, tx \ge t_0$, (3.3) holds with $\epsilon = 1$ and this δ. Fix $0 < T < \infty$. Let N be such that for $n \ge N$, $b(n/v_n) > 2t_0 \vee t_0/T$. Thus, there is $C < \infty$, whose value may change from line to line, depending only on T, so that for $n \ge N$ and $x \ge T$,

$$\left|\frac{P(H > b(n/v_n)x)}{P(H > b(n/v_n))} - x^{-\alpha}\right| \le CA(b(n/v_n))x^{-\alpha+\rho\alpha+\delta} \le CA(b(n/v_n))x^{-\alpha}$$

the second inequality following since $\rho\alpha + \delta < 0$, and similarly

$$\begin{aligned} &\sup_{T \le x < \infty}\left|\frac{P(H > b(n/v_n)x)}{P(H > (1-\varepsilon_n)b(n/v_n))} - x^{-\alpha}(1-\varepsilon_n)^{\alpha}\right| \\ \le\ & CA((1-\varepsilon_n)b(n/v_n))\left(\frac{x}{1-\varepsilon_n}\right)^{-\alpha} \\ \le\ & CA(b(n/v_n))x^{-\alpha}\,. \end{aligned}$$

Since

$$(1-\varepsilon_n)^{\alpha} - 1 = O(\varepsilon_n) = O(A(b(n/v_n)))\,,$$

it follows that there is (a possibly different) $C < \infty$ so that for all $x \ge T$,

$$\left|\frac{n}{v_n}P(H > b(n/v_n)x) - x^{-\alpha}\right| \le CA(b(n/v_n))x^{-\alpha}\,.$$

This in view of (3.4) shows that assumptions D and E hold.

Finally, we show that Assumption A holds. By (3.2), it follows that

$$1 - \alpha(1-\beta) > 0\,.$$

Let $p > 0$ be such that

$$\frac{\alpha(1-\beta)}{p} < 1 - \alpha(1-\beta)\,.$$

This choice of p ensures that

$$\frac{\alpha(2-\beta)}{p} < 1 - \alpha\left(1 - \beta - \frac{1}{p}\right)\,. \tag{3.5}$$

Note that $xP(H > x)^{1-\beta-1/p}$ is regularly varying with index $1 - \alpha(1-\beta-1/p)$ and $P(H > x)^{-(2-\beta)/p}$ is regularly varying with index $\alpha(2-\beta)/p$. Thus, by (3.5) it follows that

$$M_n P(H > M_n)^{1-\beta-1/p} \gg P(H > M_n)^{-(2-\beta)/p} \gg n^{1/p}\,,$$

the last inequality following from Assumption C. Thus

$$n^{1/p}P(H > M_n)^{1/p}M_n^{-1} \ll P(H > M_n)^{1-\beta}\,.$$

Let (ε_n) be such that

$$n^{1/p}P(H>M_n)^{1/p}M_n^{-1}\ll\varepsilon_n\ll P(H>M_n)^{1-\beta}\,.$$

Clearly, (2.1) holds with this choice of (ε_n). For (2.2), note that since $EL^p<\infty$,

$$nP(H>M_n)P(L>\varepsilon_nM_n)=O\left(nP(H>M_n)\varepsilon_n^{-p}M_n^{-p}\right)=o(1)\,.$$

This shows (2.2). Finally, for (2.3), choose $\delta>0$ so that $\rho\alpha+\delta<0$. Let t_0 be such that (3.3) holds with this δ and $\epsilon=1$. Thus, as $n\longrightarrow\infty$,

$$\begin{aligned}\left|\frac{l(\gamma M_n(1+\varepsilon_n))}{l(\gamma M_n)}-1\right| &= O\left(\left|\frac{P(H>\gamma M_n(1+\varepsilon_n))}{P(H>\gamma M_n)}-(1+\varepsilon_n)^{-\alpha}\right|\right)\\ &= O\left(A(M_n)M_n^{-\alpha+\rho\alpha+\delta}\right)\\ &= o\left(P(H>M_n)^{1-\beta}\right)\,,\end{aligned}$$

the last step following from the observations that

$$\begin{aligned}&P(H>M_n)^{-(1-\beta)}A(M_n)M_n^{-\alpha+\rho\alpha+\delta}\\ &=\left\{M_n^{-\alpha}P(H>M_n)^{-(1-\beta)}\right\}M_n^{\rho\alpha+\delta}A(M_n)\end{aligned}$$

and that each of the three terms on the right hand side go to zero. This shows that Assumption A holds and thus completes the proof. □

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