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Inverse semigroups and the Cuntz-Li algebras

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INVERSE SEMIGROUPS AND THE CUNTZ-LI ALGEBRAS

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Abstract. In this paper, we apply the theory of inverse semigroups to the C^*-algebra $U[\mathbb{Z}]$ considered in [Cun08]. We show that the C^*-algebra $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries. We explicity identify the groupoid $\mathcal{G}_{tight}$ associated to the inverse semigroup and show that $\mathcal{G}_{tight}$ is exactly the same groupoid obtained in [CL10].

1. Introduction

Ever since the appearance of the Cuntz algebras O_n and the Cuntz-Krieger algebras O_A there has been a great deal of interest in understanding the structure of C^*-algebras generated by partial isometries. The theory of graph C^*-algebras owes much to these examples. It has now been well known that these algebras admit a groupoid realisation and the groupoid turns out to be r-discrete. Another object that is closely related with an r-discrete groupoid is that of an inverse semigroup. The relationship between r-discrete groupoids and inverse semigroups was already clear from [Ren80].

An inverse semigroup S is a semigroup such that for every $s \in S$, there exists a unique $s^* \in S$ for which $s^*ss^* = s^*$ and $ss^*s = s$. The universal example of an inverse semigroup is the semigroup of partial bijections on a set. Just like one can associate a C^*-algebra to a group , one can associate a universal C^*-algebra related with an inverse semigroup S and is denoted $C^*(S)$. This universal C^*-algebra captures the representations of the inverse semigroup (as partial isometries on a Hilbert space). One can canonically associate an r-discrete groupoid $\mathcal{G}_S$ to an inverse semigroup S such that the C^*-algebra of the groupoid $\mathcal{G}_S$ coincides with $C^*(S)$. For a more detailed account of inverse semigroups and r-discrete groupoids, we refer to [Pat99] and [Exe08].

Recently, Cuntz and Li in [CL10] has introduced a C^*-algebra associated to every integral domain with only finite quotients. Earlier in [Cun08], Cuntz considered the integral domain $\mathbb{Z}$. Let R be an integral domain with only finite quotients. Then the universal algebra $U[R]$ is the universal C^*-algebra generated by a set of unitaries $\{u^n : n \in R\}$ and a set of partial isometries $\{s_m : m \in R^{\times}\}$ satisfying certain relations. In [CL10], it was proved that $U[R]$ is simple and purely infinite. A concrete realisation of $U[R]$ can be obtained by representing s_m and u^n on

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$\ell^2(R)$ by

$$s_m \to S_m : \delta_r \to \delta_{rm}$$
$$u^n \to U^n : \delta_r \to \delta_{r+n}$$

Then $U[R]$ is isomorphic to the C^*-algebra generated by S_m and U^n (by the simplicity of $U[R]$). The operator S_m is implemented by the multiplication by m (an injection) and U^n is implemented by the addition by n(a bijection). Thus it is immediately clear that $U[R]$ is generated by an inverse semigroup of partial isometries. Thus the theory of inverse semigroups should explain some of the results obtained by Cuntz and Li in [CL10]. The purpose of this paper is to obtain the groupoid realisation (obtained in [CL10]) by using the theory of inverse semigroups. We spell out the details only for the case $R = \mathbb{Z}$ as the analysis for general integral domains with finite quotients is similar. We should also remark that alternate approaches to the Cuntz-Li algebras were considered in [BE10] and in [KLQ10]. The main point we want to stress is if one uses the language of inverse semigroups one can obtain a groupoid realisation systematically without having to guess anything about the structure of the Cuntz-Li algebras.

Now we indicate the organisation of this paper. In Section 2, the definition of $U[\mathbb{Z}]$ is recalled and we show that $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries which we denote it by T. In Section 3, we recall the notion of tight representations of an inverse semigroup, a notion introduced by Exel in [Exe08]. We show that the identity representation of T in $U[\mathbb{Z}]$ is in fact tight, and show that $U[\mathbb{Z}]$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{tight}$ (considered in [Exe08]) associated to T. In Sections 4 and 5, we explicity identify the groupoid $\mathcal{G}_{tight}$ which turns out to be exactly the groupoid considered in [CL10]. In Section 6, we show that $U[\mathbb{Z}]$ is simple. In section 7, we digress a bit to explain the connection between Crisp and Laca's boundary relations and Exel's tight representations of Nica's inverse semigroup. In the final Section, we give a few remarks of how to adapt the analysis carried out in Sections $1-6$ for a general integral domain. A bit of notation: For non-zero integers m and n, we let $[m,n]$ to denote the lcm of m and n and (m,n) to denote the gcd of m and n. For a ring R, $R^\times$ denotes the set of non-zero elements in R.

2. The Regular C^*-algebra associated to $\mathbb{Z}$

Definition 2.1 ([Cun08]). *Let $U[\mathbb{Z}]$ be the universal C^*-algebra generated by a set of unitaries $\{u^n : n \in \mathbb{Z}\}$ and a set of isometries $\{s_m : m \in \mathbb{Z}^\times\}$ satisfying the following relations.*

$$s_m s_n = s_{mn}$$
$$u^n u^m = u^{n+m}$$
$$s_m u^n = u^{mn} s_m$$
$$\sum_{n \in \mathbb{Z}/(m)} u^n e_m u^{-n} = 1$$

where e_m denotes the final projection of s_m.

Remark 2.2. *Let χ be a character of the discrete multiplicative group $\mathbb{Q}^\times$. Then the universal property of the C^*-algebra $U[\mathbb{Z}]$ ensures that there exists an automorphism α_χ of the algebra $U[\mathbb{Z}]$ such that $\alpha_\chi(s_m) = \chi(m)s_m$ and $\alpha_\chi(u^n) = u^n$. This action of the character group of the multiplicative group $\mathbb{Q}^\times$ was considered in* [CL10].

For $m \neq 0$ and $n \in \mathbb{Z}$, Consider the operators S_m and U^n defined on $\ell^2(\mathbb{Z})$ as follows:

$$S_m(\delta_r) = \delta_{rm}$$
$$U^n(\delta_r) = \delta_{r+n}$$

Then $s_m \to S_m$ and $u^n \to U^n$ gives a representation of the universal C^*-algebra $U[\mathbb{Z}]$ called the regular representation and its image is denoted by $U_r[\mathbb{Z}]$. We begin with a series of Lemmas (highly inspired and adapted from [Cun08] and from [CL10]) which will be helpful in proving that $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries.

Lemma 2.3. *For every $m, n \neq 0$, one has $e_m = \sum_{k\in\mathbb{Z}/(n)} u^{mk} e_{mn} u^{-mk}$.*

Proof. One has

$$\begin{aligned} e_m &= s_m s_m^* \\ &= s_m\Big(\sum_{k\in\mathbb{Z}/(n)} u^k e_n u^{-k}\Big)s_m^* \\ &= \sum_{k\in\mathbb{Z}/(n)} s_m u^k s_n s_n^* u^{-k} s_m^* \\ &= \sum_{k\in\mathbb{Z}/(n)} u^{km} s_m s_n s_n^* s_m^* u^{-km} \\ &= \sum_{k\in\mathbb{Z}/(n)} u^{km} s_{mn} s_{mn}^* u^{-km} \\ &= \sum_{k\in\mathbb{Z}/(n)} u^{km} e_{mn} u^{-km}. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *For every $m, n \neq 0$, one has $e_m e_n = e_{[m,n]}$ where $[m, n]$ denotes the least common multiple of m and n.*

Proof. Let $c := [m, n]$ be the lcm of m and n. Then $c = am = bn$ for some a, b. Now from Lemma 2.3, it follows that

$$e_m e_n = \sum_{r\in\mathbb{Z}/(a), s\in\mathbb{Z}/(b)} u^{mr} e_c u^{-mr} u^{ns} e_c u^{-ns}$$

The product $u^{mr}e_cu^{-mr}u^{ns}e_cu^{-ns}$ survives if and only if $mr \equiv ns \mod c$. But the only choice for such an r and an s is when $r \equiv 0 \mod a$ and $s \equiv 0 \mod b$. [Reason : Suppose there exists r and s such that $mr \equiv ns \mod c$. Then $\frac{mr-ns}{c}$ is an integer. That is $\frac{r}{a} - \frac{s}{b}$ is an integer. Multiplying by b, one has that $\frac{br}{a} - s$ and hence $\frac{br}{a}$ is an integer. But a and b are relatively prime. Hence a divides r. Similarly b divides s]. Thus $e_me_n = e_c$. This completes the proof. □

Lemma 2.5. *Suppose $r \neq s$ in $\mathbb{Z}/(d)$ then the projections $u^re_mu^{-r}$ and $u^se_nu^{-s}$ are orthogonal where d is the gcd of m and n.*

Proof. First note that $e_du^{-r}u^se_du^{-s}u^r = 0$. Hence $e_du^{-r}u^se_d = 0$. Now note that

$$\begin{aligned} u^re_mu^{-r}u^se_nu^{-s} &= u^re_me_du^{-r}u^se_de_nu^{-s} \quad \text{[by Lemma 2.4]} \\ &= u^re_m(e_du^{-r}u^se_d)e_nu^{-s} \\ &= 0 \end{aligned}$$

This completes the proof. □

Lemma 2.6. *Let $m, n \neq 0$ be given. Let $d = (m, n)$ and $c = [m, n]$. Suppose $r \equiv s \mod d$. Let k be such that $k \equiv r \mod m$ and $k \equiv s \mod n$. Then $u^re_mu^{-r}u^se_nu^{-s} = u^ke_cu^{-k}$.*

Proof. First note that $u^re_mu^{-r} = u^ke_mu^{-k}$ and $u^se_nu^{-s} = u^ke_nu^{-k}$. The result follows from Lemma 2.4. □

Lemma 2.7. *For $m, n \neq 0$, one has $s_m^*e_ns_m = e_{n'}$ where $n' := \frac{n}{(n,m)}$.*

Proof. First note that without loss of generality, we can assume that m and n are relatively prime. Otherwise write $m := m_1d$ and $n := n_1d$ where d is the gcd of m and n. Then $(m_1, n_1) = 1$ and

$$\begin{aligned} s_m^*e_ns_m &= s_{m_1}^*s_d^*s_ds_{n_1}s_{n_1}^*s_d^*s_ds_{m_1} \\ &= s_{m_1}^*e_{n_1}s_{m_1} \end{aligned}$$

So now assume m and n are relatively prime. Observe that $s_m^*e_ns_m$ is a selfadjoint projection. For $s_m^*e_ns_ms_m^*e_ns_m = s_m^*e_ne_ms_m = s_m^*e_me_ns_m = s_m^*e_ns_m$. Again,

$$\begin{aligned} (s_m^*e_ns_m)^2 &= s_m^*e_ne_ms_m \\ &= s_m^*e_{mn}s_m \text{ [by Lemma 2.4]} \\ &= s_m^*s_ms_ns_n^*s_m^*s_m \\ &= e_n \end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let $m, n \neq 0$ and $k \in \mathbb{Z}$ be given. If (m, n) does not divide k then one has $s_m^*u^ke_nu^{-k}s_m = 0$.*

Proof. It is enough to show that $x := e_n u^{-k} s_m$ vanishes. Thus it is enough to show that $xx^* = e_n u^{-k} e_m u^k e_n$. Now Lemma 2.5 implies that $xx^* = 0$. This completes the proof. □

Lemma 2.9. *Let $m, n \neq 0$ and $k \in \mathbb{Z}$ be given. Suppose that $d := (m, n)$ divides k. Choose an integer r such that $mr \equiv k \mod n$. Then $s_m^* u^k e_n u^{-k} s_m = u^r e_{n_1} u^{-r}$ where $n_1 = \frac{n}{d}$.*

Proof. Now observe that $u^k e_n u^{-k} = u^{mr} e_n u^{-mr}$. Hence one has

$$\begin{aligned} s_m^* u^k e_n u^{-k} s_m &= s_m^* u^{mr} e_n u^{-mr} s_m \\ &= u^r s_m^* e_n s_m u^{-r} \\ &= u^r e_{n_1} u^{-r} \text{ [by Lemma 2.7]} \end{aligned}$$

This completes the proof. □

Remark 2.10. *Let $P := \{u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z}\} \cup \{0\}$. Then the above observations show that P is a commutative semigroup of projections which is invariant under the map $x \to s_m^* x s_m$.*

The proof of the following proposition is adapted from [CL10].

Proposition 2.11. *Let $T := \{s_m^* u^n e_k u^{n'} s_{m'} : m, m', k \neq 0, n, n' \in \mathbb{Z}\} \cup \{0\}$. Then T is an inverse semigroup of partial isometries. Let $P := \{u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z}\} \cup \{0\}$. Then the set of projections in T coincide with P. Also the linear span of T is dense in $U[\mathbb{Z}]$.*

Proof. The fact that T is closed under multiplication follows from the following calculation.

$$\begin{aligned} s_m^* u^n e_r u^{-n'} s_{m'} s_k^* u^\ell e_s u^{-\ell'} s_{k'} &= s_m^* u^n e_r u^{-n'} s_{m'} s_{m'}^* s_k^* s_{m'} u^\ell e_s u^{-\ell'} s_{k'} \\ &= s_m^* u^{n-n'} u^{n'} e_r u^{-n'} e_{m'} s_k^* s_{m'} u^\ell e_s u^{-\ell} u^{\ell-\ell'} s_{k'} \\ &= s_m^* u^{n-n'} \tilde{e} e_{m'} s_k^* s_{m'} \tilde{f} u^{\ell-\ell'} s_{k'} \text{ [where } \tilde{e} = u^{n'} e_r u^{-n'} \text{ and } \tilde{f} = u^\ell e_s u^{-\ell}] \\ &= s_m^* u^{n-n'} s_k^* (s_k \tilde{e} s_k^*)(s_k e_{m'} s_k^*)(s_{m'} \tilde{f} s_{m'}^*) s_{m'} u^{\ell-\ell'} s_{k'} \\ &= s_{mk}^* u^{kn-kn'} p u^{\ell m' - \ell' m'} s_{k'm'} \text{ [where } p := (s_k \tilde{e} s_k^*)(s_k e_{m'} s_k^*)(s_{m'} \tilde{f} s_{m'}^*) \in P \text{]} \end{aligned}$$

Thus we have shown that T is closed under multiplication. Clearly T is closed under the involution $*$. Thus the linear span of T is a $*$ algebra containing s_m and u^n for every $m \neq 0$ and $n \in \mathbb{Z}$. Hence the linear span of T is dense in $U[\mathbb{Z}]$.

Now we show that every element of T is a partial isometry. Let $v := s_m^* u^n e_k u^{n'} s_{m'}$ be given. Now,

$$\begin{aligned} vv^* &= s_m^* u^n e_k u^{n'} s_{m'} s_{m'}^* u^{-n'} e_k u^{-n} s_m \\ &= s_m^* u^n (e_k u^{n'} e_{m'} u^{-n'} e_k) u^{-n} s_m \\ &= s_m^* u^n e u^{-n} s_m \text{ [where } e := (e_k u^{n'} e_{m'} u^{-n'} e_k) \in P \text{]} \end{aligned}$$

Now it follows from Remark 2.10 that $vv^* \in P$. It also shows that the set of projections in T coincides with P. This completes the proof. □

The following equality will be used later. Let us isolate it now.

$$s_{m_1}^* u^{k_1} s_{n_1} s_{m_2}^* u^{k_2} s_{n_2} = s_{m_1 m_2}^* u^{m_2 k_1} e_{m_2 n_1} u^{k_2 n_1} s_{n_1 n_2} \tag{2.1}$$

Remark 2.12. *We also need the following fact. If $v \in T$, let us denote its image in the regular representation by V. Observe that $v \neq 0$ if and only if $V \neq 0$. This is clear for projections in T. Now let $v \in T$ be a non-zero element. Then $vv^* \in P$ is non-zero. Thus $VV^* \neq 0$ which implies $V \neq 0$.*

3. Tight representations of an inverse semigroup

Let us recall the notion of tight characters and tight representations from [Exe08].

Definition 3.1. *Let S be an inverse semigroup with 0. Denote the set of projections in S by E. A character for E is a map $x : E \to \{0,1\}$ such that*

1. *the map x is a semigroup homomorphism, and*
2. *$x(0) = 0$.*

We denote the set of characters of E by $\hat{E}_0$. We consider $\hat{E}_0$ as a locally compact Hausdorff topological space where the topology on $\hat{E}_0$ is the subspace topology induced from the product topology on $\{0,1\}^E$.

For a character x of E, let $A_x := \{e \in E : x(e) = 1\}$. Then A_x is a nonempty set satisfying the following properties.

1. The element $0 \notin A_x$.
2. If $e \in A_x$ and $f \geq e$ then $f \in A_x$.
3. If $e, f \in A_x$ then $ef \in A_x$.

Any nonempty subset A of E for which $(1), (2)$ and (3) are satisfied is called a filter. Moreover if A is a filter then the indicator function 1_A is a character. Thus there is a bijective correspondence between the set of characters and filters. A filter is called an ultrafilter if it is maximal. We also call a character x maximal or an ultrafilter if its support A_x is maximal. The set of maximal characters is denoted by $\hat{E}_\infty$ and its closure in $\hat{E}_0$ is denoted by $\hat{E}_{tight}$.

The following characterization of maximal characters will be extremely useful for us and we refer to [Exe09] for a proof. Let E be an inverse semigroup of projections. Let $e, f \in E$. We say that f intersects e if $fe \neq 0$.

Lemma 3.2. *Let E be an inverse semigroup of projections with 0 and x be a character of E. Then the following are equivalent.*

1. *The character x is maximal.*
2. *The support A_x contains every element of E which intersects every element of A_x.*

Corollary 3.3. *Let A be a unital C^*-algebra and $E \subset A$ be an inverse semigroup of projections containing $\{0,1\}$. Suppose that E contains a finite set $\{e_1, e_2, \cdots, e_n\}$ of mutually orthogonal projections such that $\sum_{i=1}^n e_i = 1$. Then for every maximal character x of E, there exists a unique e_i for which $x(e_i) = 1$.*

Proof. The uniqueness of e_i is clear as the projections $e_1, e_2, \cdots, e_n$ are orthogonal. Now to show the existence of an e_i in A_x, we prove by contradiction. Assume that $e_i \notin A_x$ for every i. Then by Lemma 3.2, we have that for every i, there exists an $f_i \in A_x$ such that $e_i f_i = 0$. Let $f = \prod f_i$. Then $f \in A_x$ and thus nonzero and also $fe_i = 0$ for every i. As $\sum_i e_i = 1$, this forces $f = 0$. Thus we have a contradiction. □

Let us recall the notion of tight representations of semilattices from [Exe08] and from [Exe09]. The only semilattice we consider is that of an inverse semigroup of projections or in otherwords the idempotent semilattice of an inverse semigroup. Also our semilattice contains a maixmal element 1. First let us recall the notion of a cover from [Exe08].

Definition 3.4. *Let E be an inverse semigroup of projections containing $\{0,1\}$ and Z be a subset of E. A subset F of Z is called a cover for Z if given a non-zero element $z \in Z$ there exists an $f \in F$ such that $fz \neq 0$. A cover F of Z is called a finite cover if F is finite.*

The following definition is actually Proposition 11.8 in [Exe08]

Definition 3.5. *Let E be an inverse semigroup of projections containing $\{0,1\}$. A representation $\sigma : E \to \mathcal{B}$ of the semilattice E in a Boolean algebra $\mathcal{B}$ is said to be tight if given $e \neq 0$ in E and for every finite cover F of the interval $[0,e] := \{x \in E : x \leq e\}$, one has $\sup_{f \in F} \sigma(f) = \sigma(e)$.*

Let A be a unital C^* algebra and S be an inverse semigroup containing $\{0,1\}$. Let $\sigma : S \to A$ be a unital representation of S as partial isometries in A. Let $\sigma(C^*(E))$ be the C^*-subalgebra in A generated by $\sigma(E)$. Then $\sigma(C^*(E))$ is a unital, commutative C^*-algebra and hence the set of projections in it is a Boolean algebra which we denote by $\mathcal{B}_{\sigma(C^*(E))}$. We say the representation σ is **tight** if the representation $\sigma : E \to \mathcal{B}_{\sigma(C^*(E))}$ is **tight**.

Lemma 3.6. *Let X be a compact metric space and $E \subset C(X)$ be an inverse semigroup of projections containing $\{0,1\}$. Suppose that for every finite set of projections $\{f_1, f_2, \cdots, f_m\}$ in E, there exists a finite set of mutually orthogonal non-zero projections $\{e_1, e_2, \cdots, e_n\}$ in E and a matrix (a_{ij}) such that*

$$\sum_{i=1}^n e_i = 1$$

$$f_i = \sum_j a_{ij} e_j.$$

Then the identity representation of E in $C(X)$ is tight.

Proof. Let $e \in E\backslash\{0\}$ be given and let F be a finite cover for the interval $[0.e]$. Without loss of generality, we can assume that $e = 1$ (Just cut everything down by e). Let $F := \{f_1, f_2, \cdots, f_m\}$. Then by the hypothesis there exists a finite set of mutually orthogonal projections $\{e_1, e_2, \cdots, e_n\}$ and a matrix (a_{ij}) such that $f_i = \sum_j a_{ij}e_j$ and $\sum_i e_i = 1$. For a given j, let $A_j := \{i : a_{ij} \neq 0\}$. Since F covers $C(X)$, it follows that for every j, A_j is nonempty. In otherwords, given j, there exists an i such that $f_i \geq e_j$. Thus $f := \sup_i f_i \geq e_j$ for every j. Hence $f \geq \sup_j e_j = 1$. This completes the proof. □

In the next proposition, T denotes the inverse semigroup associated to $U[\mathbb{Z}]$ in Proposition 2.11.

Proposition 3.7. *The identity representation of T in $U[\mathbb{Z}]$ is tight.*

Proof. We apply Lemma 3.6. Let $\{u^{r_1}e_{m_1}u^{-r_1}, u^{r_2}e_{m_2}u^{-r_2}, \cdots, u^{r_k}e_{m_k}u^{-r_k}\}$ be a finite set of non-zero projections in P. By Lemma 2.3, it follows that each $f_i := u^{r_i}e_{m_i}u^{-r_i}$ is a linear combination of $\{u^s e_c u^{-s} : s \in \mathbb{Z}/(c)\}$ where c is the lcm of $m_1, m_2, \cdots, m_k$. Then Lemma 3.6 implies that the identity representation of T in $U[\mathbb{Z}]$ is tight. This completes the proof. □.

Now we will show that the C^*-algebra of the groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T is isomorphic to the algebra $U[\mathbb{Z}]$. First let us recall the construction of the groupoid $\mathcal{G}_{tight}$ considered in [Exe08]. Let S be an inverse semigroup with 0 and let E denote its set of projections. Note that S acts on $\hat{E}_0$ partially. For $x \in \hat{E}_0$ and $s \in S$, define $(x.s)(e) = x(ses^*)$. Then

- The map $x.s$ is a semigroup homomorphism, and
- $(x.s)(0) = 0$.

But $x.s$ is nonzeo if and only if $x(ss^*) = 1$. For $s \in S$, define the domain and range of s as Let S be an inverse semigroup with 0 and let E denote its set of projections. Note that S acts on $\hat{E}_0$ partially. For $x \in \hat{E}_0$ and $s \in S$, define $(x.s)(e) = x(ses^*)$. Then

- The map $x.s$ is a semigroup homomorphism, and
- $(x.s)(0) = 0$.

But $x.s$ is nonzeo if and only if $x(ss^*) = 1$. For $s \in S$, define the domain and range of s as

$$D_s := \{x \in \hat{E}_0 : x(ss^*) = 1\}$$
$$R_s := \{x \in \hat{E}_0 : x(s^*s) = 1\}$$

Note that both D_s and R_s are compact and open. Moreover s defines a homoemorphism from D_s to R_s with s^* as its inverse. Also observe that $\hat{E}_{tight}$ is invariant under the action of S.

Consider the transformation groupoid $\Sigma := \{(x, s) : x \in D_s\}$ with the composition and the inversion being given by:

$$(x, s)(y, t) := (x, st) \text{ if } y = x.s$$
$$(x, s)^{-1} := (x.s, s^*)$$

Define an equivalence relation $\sim$ on Σ as $(x,s) \sim (y,t)$ if $x = y$ and if there exists an $e \in E$ such that $x \in D_e$ for which $es = et$. Let $\mathcal{G} = \Sigma/\sim$. Then $\mathcal{G}$ is a groupoid as the product and the inversion respects the equivalence relation $\sim$. Now we describe a toplogy on $\mathcal{G}$ which makes $\mathcal{G}$ into a topological groupoid.

For $s \in S$ and U an open subset of D_s, let $\theta(s,U) := \{[x,s] : x \in U\}$. We refer to [Exe08] for the proof of the following two propositions. We denote $\theta(s, D_s)$ by θ_s. Then θ_s is homeomorphic to D_s and hence is compact, open and Hausdorff.

Proposition 3.8. *The collection $\{\theta(s,U) : s \in S, U \text{ open in } D_s\}$ forms a basis for a topology on $\mathcal{G}$. The groupoid $\mathcal{G}$ with this topology is a topological groupoid whose unit space can be identified with $\hat{E}_0$. Also one has the following.*

1. *For $s,t \in S$, $\theta_s\theta_t = \theta_{st}$,*
2. *For $s \in S$, $\theta_s^{-1} = \theta_{s^*}$, and*
3. *The set $\{1_{\theta_s} : s \in T\}$ generates the C^* algebra $C^*(\mathcal{G})$.*

We define the groupoid $\mathcal{G}_{tight}$ to be the reduction of the groupoid $\mathcal{G}$ to $\hat{E}_{tight}$. In [Exe08], it is shown that the representation $s \to 1_{\theta_s} \in C^*(\mathcal{G}_{tight})$ is tight and any tight representation factors through this universal one.

Proposition 3.9. *Let T be the inverse semigroup associated to $U[\mathbb{Z}]$ in Proposition 2.11. Let $\mathcal{G}_{tight}$ be the tight groupoid associated to T. Then $U[\mathbb{Z}]$ is isomorphic to $C^*(\mathcal{G}_{tight})$.*

Proof. Let t_m, v^n denote the images of s_m, u^n in $C^*(\mathcal{G}_{tight})$. The universality of the C^*-algebra $C^*(\mathcal{G}_{tight})$ together with Proposition 3.7 implies that there exists a homomorphism $\rho : C^*(\mathcal{G}_{tight}) \to U[\mathbb{Z}]$ such that $\rho(t_m) = s_m$ and $\rho(v^n) = u^n$.

Note that the mutually orthogonal set of projections $\{u^r e_m u^{-r} : r \in \mathbb{Z}/(m)\}$ cover T. Since the representation of T in $C^*(\mathcal{G}_{tight})$ is tight, it follows that $\sum_r v^r t_m t_m^* v^{-r} = 1$. Now the universal property of $U[\mathbb{Z}]$ implies that there exists a homomorphism $\sigma : U[\mathbb{Z}] \to C^*(\mathcal{G}_{tight})$ such that $\sigma(s_m) = t_m$ and $\sigma(u^n) = v^n$. Now it is clear that ρ and σ are inverses of each other. This completes the proof. □

In the next two sections, we identify the groupoid $\mathcal{G}_{tight}$ explicitly.

4. Tight characters of the inverse semigroup T

In this section, we determine the tight characters of the inverse semigroup T defined in Proposition 2.11. Let us recall a few ring theoretical notions. We denote the set of strictly positive integers by $\mathbb{N}^+$. Consider the directed set $(\mathbb{N}^+, \leq)$ where we say $m \leq n$ if $m|n$. If $m|n$ then there exists a natural map from $\mathbb{Z}/(n)$ to $\mathbb{Z}/(m)$. The inverse limit of this system is called the profinite completion of $\mathbb{Z}$ and is denoted $\hat{\mathbb{Z}}$. In other words,

$$\hat{\mathbb{Z}} := \{(r_m) \in \prod_{m \in \mathbb{N}^+} \mathbb{Z}/(m) : r_{mk} \cong r_m \; mod \; m\}$$

Also $\hat{\mathbb{Z}}$ is a compact ring with the subspace topology induced by the product topology on $\prod \mathbb{Z}/(m)$. Also $\mathbb{Z}$ embedds naturally in $\hat{\mathbb{Z}}$. We also need the easily verifiable fact that the kernel of the m^{th} projection $r = (r_m) \to r_m$ is in fact $m\hat{\mathbb{Z}}$.

For $r \in \hat{\mathbb{Z}}$, define a character $\xi_r : P \to \{0, 1\}$ by the following formula:

$$\xi_r(u^n e_m u^{-n}) := \delta_{r_m,n}$$
$$\xi_r(0) := 0$$

In the above formula, the Dirac-delta function is over the set $\mathbb{Z}/(m)$. Thus $\delta_{r_m,n} = 1$ if and only if $r_m \equiv n \mod m$.

Proposition 4.1. *The map $r \to \xi_r$ is a topological isomorphism from $\hat{\mathbb{Z}}$ to $\hat{P}_{tight}$*

Proof. First let us check that for $r \in \hat{\mathbb{Z}}$, ξ_r is in fact a character and is maximal. Consider an element $r \in \hat{\mathbb{Z}}$. Let $e := u^{n_1} e_{m_1} u^{-n_1}$ and $f := u^{n_2} e_{m_2} u^{-n_1}$ be given. Let $d := (m_1, m_2)$ and $c := [m_1, m_2]$. Suppose $\xi_r(e) = \xi_r(f) = 1$. Then $r_{m_1} \equiv n_1 \mod m_1$ and $r_{m_2} \equiv n_2 \mod m_2$. Moreover, $r_c \equiv r_{m_i} \mod m_i$ for $i = 1, 2$. Thus $ef = u^{r_c} e_c u^{-r_c}$ by Lemma 2.6 Hence by definition $\xi_r(ef) = 1$. Now suppose $\xi_r(e) = 1$ and $e \leq f$. Then by Lemma 2.5 and Lemma 2.6, it follows that m_2 divides m_1 and $r_{m_2} \equiv r_{m_1} \equiv n_1 \equiv n_2 \mod m_2$. Hence $\xi_r(f) = 1$. By definition 0 is not in the support of ξ_r. Thus we have shown that the support of ξ_r is a filter or in other words ξ_r is a character.

Now we claim ξ_r is maximal. This follows from the observation that for every $m \in \mathbb{N}^+$, the set of projections $\{u^n e_m u^{-n} : n \in \mathbb{Z}/(m)\}$ are mutually orthogonal. Thus if ξ is a character then for every m there exists at most one r_m for which $\xi(u^{r_m} e_m u^{-r_m}) = 1$. This implies that if ξ is a character which contains the support of ξ_r then $\xi = \xi_r$.

Now let ξ be a maximal character of P. Then by Corollary 3.3 and by the observation in the previous paragraph, it follows that for every m there exists a unique r_m such that $\xi(u^{r_m} e_m u^{-r_m}) = 1$. Now let k be given. Since both $u^{r_m} e_m u^{-r_m}$ and $u^{r_{mk}} e_{mk} u^{-r_{mk}}$ belong to the support of ξ, it follows that the product $u^{r_m} e_m u^{-r_m} u^{r_{mk}} e_{mk} u^{-r_{mk}}$ does not vanish. Then by Lemma 2.5, it follows that $r_{mk} \equiv r_m \mod m$. Thus $r = (r_m) \in \hat{\mathbb{Z}}$ and the support of ξ_r is contained in the support of ξ. Thus again by the observation in the preceeding paragraph, it follows that $\xi = \xi_r$.

It is clear from the definition that the map $r \to \xi_r$ is one-one and continuous. As $\hat{\mathbb{Z}}$ is compact, it follows that the range of the map $r \to \xi_r$ which is $\hat{P}_\infty$ is also compact. Hence $\hat{P}_\infty = \hat{P}_{tight}$. Thus we have shown that $r \to \xi_r$ is a one-one and onto continuous map from $\hat{\mathbb{Z}}$ to $\hat{P}_{tight}$. Since $\hat{\mathbb{Z}}$ is compact, it follows that the above map is in fact a homeomorphism. This completes the proof. □

From now on we will simply write $r(e)$ in place of $\xi_r(e)$ if $r \in \hat{\mathbb{Z}}$ and $e \in P$.

5. The groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T

Let us recall a few ring theoretical constructions. Consider the directed set $(\mathbb{N}^+, \leq)$ where the partial order $\leq$ is defined by $m \leq n$ if m divides n. For $m \in \mathbb{N}^+$, let $\mathcal{R}_m := \hat{\mathbb{Z}}$. Let $\phi_{m\ell,m} : \mathcal{R}_m \to \mathcal{R}_{\ell m}$ be the map defined by mulitplication by ℓ. Then $\phi_{m\ell,m}$ is only an additive homomorphism and it does not preserve the multiplication. We let $\mathcal{R}$ be the inductive limit of $(\mathcal{R}_m, \phi_{m\ell,m})$. Then $\mathcal{R}$ is an abelian group and $\hat{\mathbb{Z}}$ is a subgroup of $\mathcal{R}$ via the inclusion $\mathcal{R}_1 \subset \mathcal{R}$. Note that $\mathcal{R}$ is a locally compact Hausdorff space. Moreover the group $P_{\mathbb{Q}} := \Big\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in \mathbb{Q}^\times, b \in \mathbb{Q} \Big\}$ acts on $\mathcal{R}$ by affine transformations. The action is descibed explicitly by the following formula. For $x \in \mathcal{R}_p$

$$\begin{bmatrix} 1 & 0 \\ \frac{n}{m'} & \frac{m}{m'} \end{bmatrix} x = mx + np \in \mathcal{R}_{m'p}$$

One can check that the above formula defines an action of $P_{\mathbb{Q}}$ on $\mathcal{R}$. We need the following lemma.

Lemma 5.1. *Let $a := \frac{n}{m'}$ and $b := \frac{m}{m'}$. Then $s_{m'}^* u^n s_m$ depends only on a and b.*

Proof. Suppose $\frac{n_1}{m_1'} = \frac{n_2}{m_2'}$ and $\frac{m_1}{m_1'} = \frac{m_2}{m_2'}$. Then $n_1 m_2' = n_2 m_1'$ and $m_1 m_2' = m_1' m_2$. Now, we have

$$\begin{aligned}
s_{m_1'}^* u^{n_1} s_{m_1} &= s_{m_1'}^* s_{m_2}^* s_{m_2} u^{n_1} s_{m_1} \\
&= s_{m_2'}^* s_{m_1}^* s_{m_1'}^* s_{m_1'} s_{m_2} u^{n_1} s_{m_1} \\
&= s_{m_2'}^* s_{m_1}^* s_{m_1'}^* u^{n_1 m_2 m_1'} s_{m_1'} s_{m_1} s_{m_2} \\
&= s_{m_2'}^* s_{m_1'}^* s_{m_1}^* u^{n_1 m_2' m_1} s_{m_1} s_{m_1'} s_{m_2} \\
&= s_{m_2'}^* s_{m_1'}^* u^{n_1 m_2'} s_{m_1'} s_{m_2} \\
&= s_{m_2'}^* s_{m_1'}^* u^{n_2 m_1'} s_{m_1'} s_{m_2} \\
&= s_{m_2'}^* u^{n_2} s_{m_1'}^* s_{m_1'} s_{m_2} \\
&= s_{m_2'}^* u^{n_2} s_{m_2}
\end{aligned}$$

This completes the proof. □

Remark 5.2. *The above lemma has also been used in* [BE10].

Now we explicitly identify the groupoid $\mathcal{G}_{tight}$ associated to the inverse semigroup T. When we consider transformation groupoids, we consider only right actions. Thus we let $P_{\mathbb{Q}}$ act on $\mathcal{R}$ on the right by defining $x.g = g^{-1}x$ for $x \in \mathcal{R}$ and $g \in P_{\mathbb{Q}}$. We show that that groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T is isomorphic to the restriction of the transformation groupoid

$\mathcal{R} \times P_{\mathbb{Q}}$ to the closed subset $\hat{\mathbb{Z}}$. (Here we consider $P_{\mathbb{Q}}$ as a discrete group.) Let us begin with a lemma which will be useful in the proof.

Lemma 5.3. *In $\mathcal{G}_{tight}$ one has $[(r, s_{m'}^* u^{n'} e_k u^n s_m)] = [(r, s_{m'}^* u^{n+n'} s_m)]$*

Proof. First observe that $[(r, s_{m'}^*)][(r.s_{m'}^*, u^{n'} e_k u^n s_m)] = [(r, s_{m'}^* u^{n'} e_k u^n s_m)]$. Thus it is enough to consider the case $m' = 1$. Now let $s := u^{n'} e_k u^n s_m$, $t := u^{n+n'} s_m$ and $e := u^{n'} e_k u^{-n'}$. Now observe that $ss^* := ett^*$. Hence if $r(ss^*) = 1$ then $r(tt^*) = 1$ and $r(e) = 1$. Moreover $es = et$. Thus $[(r, s)] = [(r, t)]$. This completes the proof. □.

Theorem 5.4. *Let $\phi : \mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}} \to \mathcal{G}_{tight}$ be the map defined by*

$$\phi\Big((r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix})\Big) = [(r, s_m^* u^k s_n)]$$

Then ϕ is a topological groupoid isomorphism.

Proof.

The map ϕ is well defined.

Let $(r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix})$ be an element in $\mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$. Then we have $mr - k = ns$ for some $s \in \hat{\mathbb{Z}}$. Now we need to show that $r(s_m^* u^k e_n u^{-k} s_m) = 1$. By Lemma 2.9, it follows that $s_m^* u^k e_n u^{-k} s_m = u^{r_n} e_{n_1} u^{-r_n}$ where $n_1 := \frac{n}{(n,m)}$. Thus

$$\begin{aligned} r(s_m^* u^k e_n u^{-k} s_m) &= r(u^{r_n} e_{n_1} u^{-r_n}) \\ &= \delta_{r_{n_1}, r_n} \\ &= 1 \text{ [Since } r_n = r_{n_1} \text{ in } \mathbb{Z}/(n_1)] \end{aligned}$$

Surjectivity of ϕ:

First let us show that if $[(r, s_m^* u^k s_n)] \in \mathcal{G}_{tight}$ then $\Big(r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix}\Big) \in \mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$. Consider an element $[(r, v := s_m^* u^k s_n)]$ in $\mathcal{G}_{tight}$. Then $r(vv^*) = 1$ and $vv^* := s_m^* u^k e_n u^{-k} s_m$. Now Lemma 2.8 and 2.9 implies that $(m, n)|k$. Let s be an integer such that $ms \equiv k \mod n$. Again Lemma 2.9 implies that $vv^* = u^s e_{n_1} u^{-s}$ where $n_1 := \frac{n}{(n,m)}$. Now $r(vv^*) = 1$ implies that $r_{n_1} \equiv s \mod n_1$. But $r_n \equiv r_{n_1} \mod n_1$ (as $r \in \hat{\mathbb{Z}}$). Thus we have $r_n \equiv s \mod n_1$. This in turn implies that $mr_n \equiv ms \equiv k \mod n$. Hence $mr - k \in n\hat{\mathbb{Z}}$. Hence $\Big(r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix}\Big) \in \mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$. Now the surjectivity of ϕ follows from Lemma 5.3.

Injectivity of ϕ:

Now suppose $[(r, s_{m_1}^* u^{k_1} s_{n_1})] = [(r, s_{m_2}^* u^{k_2} s_{n_2})]$. Then by definition there exists a projection of the form $e := u^{r_p} e_p u^{-r_p}$ such that $e(s_{m_1}^* u^{k_1} s_{n_1}) = e(s_{m_2}^* u^{k_2} s_{n_2}) \neq 0$. Consider a character χ

of the discrete group $\mathbb{Q}^*$. Let α_χ be the automorphism of the algebra $U[\mathbb{Z}]$ such that $\alpha_\chi(u^n) = u^n$ and $\alpha_\chi(s_m) = \chi(m)s_m$.

$$\begin{aligned}\chi(\frac{n_1}{m_1})e(s_{m_1}^*u^{k_1}s_{n_1}) &= \alpha_\chi\big(e(s_{m_1}^*u^{k_1}s_{n_1})\big)\\ &= \alpha_\chi\big(e(s_{m_2}^*u^{k_2}s_{n_2})\big)\\ &= \chi(\frac{n_2}{m_2})e(s_{m_2}^*u^{k_2}s_{n_2})\\ &= \chi(\frac{n_2}{m_2})e(s_{m_1}^*u^{k_1}s_{n_1})\end{aligned}$$

Since $e(s_{m_1}^*u^{k_1}s_{n_1}) \neq 0$, it follows that $\chi(\frac{n_1}{m_1}) = \chi(\frac{n_2}{m_2})$ for every character χ of the discrete, multiplicative group $\mathbb{Q}^*$. Thus $\frac{n_1}{m_1} = \frac{n_2}{m_2}$.

From remark 2.12, it follows that $e(s_{m_1}^*u^{k_1}s_{n_1}) = e(s_{m_2}^*u^{k_2}s_{n_2}) \neq 0$ in $U_r[\mathbb{Z}]$. Since $\frac{n_1}{m_1} = \frac{n_2}{m_2}$, it follows immediately that $\frac{k_1}{m_1} = \frac{k_2}{m_2}$. Thus we have shown that ϕ is injective.

<u>The map ϕ is a homeomorphism.</u>

First we show ϕ is continuous. Let (r_n, g_n) be a sequence in $\mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$ converging to (r, g). Since we are considering $P_{\mathbb{Q}}$ as a discrete group, we can without loss of generality assume that $g_n = g$ for every n. Then, from Lemma 4.1, it follows that $\phi(r_n, g_n)$ converges to $\phi(r, g)$.

For an open subset U of $\hat{\mathbb{Z}}$ and $g := \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix}$, consider the open set

$$\theta(U, g) := \{(r, g) : r \in U \text{ and } r.g \in \hat{\mathbb{Z}}\}.$$

Then the collection $\{\theta(U, g) : U \overbrace{\subset}^{open} \hat{\mathbb{Z}},\ g \in P_{\mathbb{Q}}\}$ forms a basis for $\mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$. Moreover $\phi(\theta(U, g)) = \theta(U, s_m^*u^k s_n)$. Hence ϕ is an open map. Thus we have shown that ϕ is a homeomorphism.

<u>ϕ is a groupoid morphism.</u>

First we show that ϕ preserves the source and range. By definition ϕ preserves the range. Let $\left(r, g := \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix}\right) \in \mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$ be given. Let $v := s_m^*u^n s_n$. Since $r.g \in \hat{\mathbb{Z}}$, it follows that there exists $t \in \hat{\mathbb{Z}}$ such that $mr - k = nt$. We need to show that $\xi_r.v = \xi_t$. (Just to keep things clear we write ξ_r for the character determined by r). It is enough to show that the support of ξ_t and that of $\xi_r.v$ coincide. But then both the characters are maximal and thus it is enough to show that the support of ξ_t is contained in the support of $\xi_r.v$. Thus, suppose that $\xi_t(u^\ell e_s u^{-\ell}) = 1$. Then $t_{ns} \equiv t_s \equiv \ell \mod s$. This implies $mr_{ns} - k \equiv nt_{ns} \equiv n\ell \mod ns$.

Thus $mr_{ns} \equiv k + n\ell \mod ns$. Let $n_1 := \frac{ns}{(ns,m)}$. Now observe that

$$\begin{aligned}(\xi_r.v)(u^\ell e_s u^{-\ell}) &= \xi_r(vu^\ell e_s u^{-\ell} v^*)\\ &= \xi_r(s_m^* u^k s_n u^\ell e_s u^{-\ell} s_n^* u^{-k} s_m)\\ &= \xi_r(s_m^* u^{k+n\ell} e_{ns} u^{-(k+n\ell)} s_m)\\ &= \xi_r(u^{r_{ns}} e_{n_1} u^{-r_{ns}}) \text{ [By Lemma 2.9]}\\ &= \delta_{r_{ns}, r_{n_1}}\\ &= 1 \text{ [Since } r_{ns} = r_{n_1} \textit{ in } \mathbb{Z}/(n_1)]\end{aligned}$$

Thus we have shown that the support of ξ_t is contained in the support of $\xi_r.v$ which in turn implies that $\xi_t = \xi_r.v$. Hence ϕ preserves the source.

Now we show that ϕ preserves multiplication. Let $\gamma_i := (r_i, \begin{bmatrix} 1 & 0 \\ \frac{k_i}{m_i} & \frac{n_i}{m_i} \end{bmatrix})$ for $i = 1, 2$. Since ϕ preserves the range and source, it follows that if γ_1 and γ_2 are composable, so do $\phi(\gamma_1)$ and $\phi(\gamma_2)$. Observe that

$$\begin{aligned}\phi(\gamma_1)\phi(\gamma_2) &= [(r_1, s_{m_1}^* u^{k_1} s_{n_1} s_{m_2}^* u^{k_2} s_{n_2}]\\ &= [r_1, s_{m_1 m_2}^* u^{m_2 k_1} e_{m_2 n_1} u^{k_2 n_1} s_{n_1 n_2}] \text{ (Eq. 2.1)}\\ &= [r_1, s_{m_1 m_2}^* u^{m_2 k_1 + n_1 k_2} s_{n_1 n_2}] \text{ (Lemma 5.3)}\\ &= \phi(\gamma_1 \gamma_2)\end{aligned}$$

It is easily verifiable that ϕ preserves inversion. This completes the proof. □

Remark 5.5. *Combining Proposition 3.9 and Theorem 8.3, we obtain that* $U[\mathbb{Z}]$ *is isomorphic to* $C^*(\mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}})$ *which is Remark 2 in page 17 of* [CL10].

6. Simplicity of $U[\mathbb{Z}]$

First we recall a few definitions from [Ren09]. Let $\mathcal{G}$ be an r-discrete, Hausdorff and locally compact topological groupoid. Let $\mathcal{G}^0$ be its unit space. We denote the source and range maps by s and r respectively. The arrows of $\mathcal{G}$ define an equivalence relation on $\mathcal{G}^0$ as follows:

$$x \sim y \text{ if there exists } \gamma \in \mathcal{G} \text{ such that } s(\gamma) = x \text{ and } r(\gamma) = y$$

A subset E of $\mathcal{G}^0$ is said to be invariant if the orbit of x is contained in E whenever $x \in E$. For $x \in \mathcal{G}^0$, define the isotropy group at x denoted $\mathcal{G}(x)$ by $\mathcal{G}(x) := \{\gamma \in \mathcal{G} : \ s(\gamma) = r(\gamma) = x\}$.

A groupoid $\mathcal{G}$ is said to be

- topologically principal if the set of $x \in \mathcal{G}^0$ for which $\mathcal{G}(x) = \{x\}$ is dense in $\mathcal{G}^0$.
- minimal if the only non-empty open invariant subset of $\mathcal{G}^0$ is $\mathcal{G}^0$.

We need the following theorem. We refer to [Ren09] for a proof.

Theorem 6.1. *Let $\mathcal{G}$ be an r-discrete, Hausdorff and locally compact topological groupoid. If $\mathcal{G}$ is topologically principal and minimal then $C^*_{red}(\mathcal{G})$ is simple.*

Proposition 6.2. *The C^*-algebra $U[\mathbb{Z}]$ is simple.*

Proof. Let $\mathcal{G}$ denote the groupoid $\mathcal{R} \times P_{\mathbb{Q}}|_{\hat{\mathbb{Z}}}$. Since the group $P_{\mathbb{Q}}$ is solvable, it is amenable and thus by Proposition 2.15 of [MR82], it follows that the full groupoid C^*-algebra $C^*(\mathcal{G})$ is isomorphic to the reduced algebra $C^*_{red}(\mathcal{G})$. Now we apply Theorem 6.1 to complete the proof.

First let us show $\mathcal{G}$ is minimal. Let U be a non-empty open invariant subset of $\mathcal{G}^0$. For $m = (m_1, m_2, \cdots, m_n) \in (\mathbb{Z}\backslash\{0\})^n$ and $k \in \mathbb{Z}$, let

$$U_{m,k} := \{r \in \hat{\mathbb{Z}} : \ r_{m_i} \equiv k \mod m_i\}$$

Then the collection $\{U_{m,k}\}$ (where m varies over $(\mathbb{Z}\backslash\{0\})^n$ (we let n vary too) and $k \in \mathbb{Z}$) is a basis for the topology on $\hat{\mathbb{Z}}$. Also observe that for a given m, $\bigcup_{k \in \mathbb{Z}} U_{m,k} = \hat{\mathbb{Z}}$. Moreover the translation matrix $\begin{bmatrix} 1 & 0 \\ k_1 - k_2 & 1 \end{bmatrix}$ maps U_{m,k_1} onto U_{m,k_2}. Now since U is non-empty and open, there exists an m and a k_0 such that $U_{m,k_0} \subset U$. But since U is invariant, it follows that $U_{m,k} \subset U$ for every $k \in \mathbb{Z}$. Thus $\bigcup_{k \in \mathbb{Z}} U_{m,k} \subset U$. This forces $U = \hat{\mathbb{Z}}$. This completes the proof. □

Now we show $\mathcal{G}$ is topologically principal. Let

$$E := \{r \in \hat{\mathbb{Z}} : r \neq 0, r_{p^i} = 0 \ \forall i, \text{ except for finitely many primes } \ p\}$$

If one identifies $\hat{\mathbb{Z}}$ with $\prod_{p \ prime} \hat{\mathbb{Z}}_p$ then it is clear that E is dense in $\hat{\mathbb{Z}}$. Now let $r \in E$ be given. We claim that $\mathcal{G}(r) = \{r\}$. Suppose $r. \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix} = r$. Then $mr - k = nr$. But $r_p = 0$ except for finitely many primes. Thus it follows that k is divisible by infinitely many primes which forces $k = 0$. Now $mr = nr$ and $r \neq 0$ implies $m = n$. Thus $\mathcal{G}(r) = \{r\}$. This proves that $\mathcal{G}$ is topologically principal. This completes the proof. □

7. Nica-covariance, tightness and boundary relations

In this section, we digress a bit to understand some of the results in [Nic92],[CL07] and in [LR10] from the point of view of inverse semigroups. Let us recall the notion of quasi-lattice ordered groups considered by Nica in [Nic92]. Let G be a discrete group and P a subsemigroup of G containing the identity e. Also assume that $P \cap P^{-1} = \{e\}$. Then P induces a left-invariant partial order $\leq$ on G defined by $x \leq y$ if and only if $x^{-1}y \in P$. The pair (G, P) is said to be quasi-lattice ordered if the following conditions are satisfied.

1. Any $x \in PP^{-1}$ has a least upper bound in P, and
2. If $s, t \in P$ have a common upper bound in P then s, t have a least upper bound.

If $s, t \in P$ have a common upper bound in P then we denote the least upper bound in P by $\sigma(s,t)$. It is easy to show that $s, t \in P$ have a common upper bound if and only if $s^{-1}t \in PP^{-1}$. Let us recall the Wiener-Hopf representation from [Nic92]. Consider the representation $W : P \to B(\ell^2(P))$ defined by

$$W(p)(\delta_a) := \delta_{pa}$$

where $\{\delta_a : a \in P\}$ denotes the canonical orthonormal basis of $\ell^2(P)$. Note that for $s \in P$, $W(s)$ is an isometry and $W(s)W(t) = W(st)$ for $s, t \in P$. For $s \in P$, let $M(s) = W(s)W(s)^*$ then

$$M(s)M(t) = \begin{cases} M(\sigma(s,t)) & \text{if } s \text{ and } t \text{ have a common upper bound in } P \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

Let $\mathcal{N} := \{W(s)W(t)^* : s, t \in P\} \cup \{0\}$. Then Equation (5) of Proposition 3.2 in [Nic92] implies that $\mathcal{N}$ is an inverse semigroup of partial isometries. The following definition is due to Nica.

Definition 7.1 ([Nic92])**.** *Let (G, P) be a quasi-lattice ordered group. An isometric representation $V : P \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (i.e. $V(t)^*V(t) = 1$ for $t \in P$, $V(e) = 1$ and $V(s)V(t) = V(st)$ for every $s, t \in P$) is said to be Nica-covariant if the following holds*

$$L(s)L(t) = \begin{cases} L(\sigma(s,t)) & \text{if } s \text{ and } t \text{ have common upper bound in } P \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

where we set $L(t) = V(t)V(t)^*$. In other words a Nica-covariant representation of (G, P) is nothing but a unital representation of the inverse semigroup $\mathcal{N}$ which sends 0 to 0.

Let us say a Nica-covariant representation is tight if the corresponding representation on $\mathcal{N}$ is tight. Now one might ask what are the tight representations of the inverse semigroup $\mathcal{N}$? We prove that tight representations are nothing but Nica-covariant representations satisfying the boundary relations considered by Laca and Crisp in [CL07]. This fact is implicit in [CL07] and it is in fact explicit if one applies Theorem 13.2 of [Exe09]. The author believes that it is worth recording this connection and we do this in the next proposition.

First let us fix a few notations. A finite subset F of P is said to cover P if given $x \in P$ there exists $y \in F$ such that x and y have a common upper bound in P. Let

$$\mathcal{F} := \{F \subset P : F \text{ is finite and covers } P\}$$

Proposition 7.2. *Let (G, P) be a quasi-lattice ordered group. Consider a Nica-covariant representation $V : P \to B(\mathcal{H})$. Then V is tight if and only if for every $F \in \mathcal{F}$, one has $\prod_{t \in F}(1 - V(t)V(t)^*) = 0$.*

Proof. Consider a Nica-covariant representation $V : P \to B(\mathcal{H})$. Suppose that V is tight. Let $F \in \mathcal{F}$ be given. Note that F covers P if and only if $\{M(t) : t \in F\}$ covers the set of projections in $\mathcal{N}$. Now the tightness of V implies that $\sup_{t \in F} V(t)V(t)^* = 1$. This is equivalent to saying that $\prod_{x \in F}(1 - V(t)V(t)^*) = 0$. Thus we have the implication '⇒'.

Let V be a Nica-covariant representation for which $\prod_{t\in F}(1 - V(t)V(t)^*) = 0$ for every $F \in \mathcal{F}$. We denote the set of projections in $\mathcal{N}$ by E. Then $E := \{M(t) : t \in P\} \cup \{0\}$. Let $\{M(t_1), M(t_2), \cdots, M(t_n)\} \subset [0, M(t)]$ be a finite cover. Then $M(t_i) \leq M(t)$ for every i. But this is equivalent to the fact that $t \leq t_i$.

We claim that $\{t^{-1}t_i : i = 1, 2, \cdots, n\}$ covers P. Let $s \in P$ be given. Then $t \leq ts$ which implies $M(ts) \leq M(t)$. Thus there exists a t_i such that $M(ts)M(t_i) \neq 0$. This implies that ts and t_i have a common upper bound in P. In other words, $(ts)^{-1}t_i = s^{-1}t^{-1}t_i \in PP^{-1}$. Thus s and $t^{-1}t_i$ have a common upper bound in P. This proves the claim.

By assumption it follows that $\prod_{i=1}^{n}(1 - L(t^{-1}t_i)) = 0$ where $L(s) := V(s)V(s)^*$. Now multiplying this equality on the left by $V(t)$ and on the right by $V(t)^*$, we get

$$\prod_{i=1}^{n}(V(t)V(t)^* - V(t)V(t^{-1}t_i)V(t^{-1}t_i)^*V(t)^*) = 0$$

$$\prod_{i=1}^{n}(V(t)V(t)^* - V(t_i)V(t_i)^*) = 0$$

But this is equivalent to $\sup_i L(t_i) = L(t)$. This completes the proof. □

Remark 7.3. *The relations* $\prod_{x\in F}(1 - V(t)V(t)^*) = 0$ *for* $F \in \mathcal{F}$ *are the boundary relations considered in* [CL07].

Let $Q_{\mathbb{N}}$ be the C^*-subalgebra of $U[\mathbb{Z}]$ generated by u and $\{s_m : m > 0\}$. In [Cun08], it was proved that $Q_{\mathbb{N}}$ is simple and purely infinite. Moreover in [Cun08], it was shown that $U[\mathbb{Z}]$ is isomorphic to a crossed product of $Q_{\mathbb{N}}$ with $\mathbb{Z}/2\mathbb{Z}$. Let

$$P_{\mathbb{N}} := \left\{ \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} : k \in \mathbb{N} \text{ and } m \in \mathbb{N}^{\times} \right\}$$

Note that $P_{\mathbb{N}}$ is a semigroup of $P_{\mathbb{Q}}$.

Remark 7.4. *In* [LR10]*, it was proved that* $(P_{\mathbb{Q}}, P_{\mathbb{N}})$ *is a quasi-lattice ordered group. Moreover it was shown in* [LR10] *that for the quasi-lattice ordered group* $(P_{\mathbb{Q}}.P_{\mathbb{N}})$ *Nica-covariance together with boundary relations is equivalent to Cuntz-Li relations and the universal* C^**-algebra made out of Nica-covariant representations satisfying the boundary relations is in fact* $Q_{\mathbb{N}}$.

8. The Cuntz-Li algebra for a general integral domain

We end this article by giving a few remarks of how to adapt the analysis in Section $1 - 6$ for a general integral domain R. Now Let R be an integral domain such that R/mR is finite for every non-zero $m \in R$. We also assume that R is countable and R is not a field.

Definition 8.1 ([CL10]). *Let $U[R]$ be the universal C^*-algebra generated by a set of unitaries $\{u^n : n \in R\}$ and a set of isometries $\{s_m : m \in R^\times\}$ satisfying the following relations.*

$$\begin{aligned} s_m s_n &= s_{mn} \\ u^n u^m &= u^{n+m} \\ s_m u^n &= u^{mn} s_m \\ \sum_{n \in R/mR} u^n e_m u^{-n} &= 1 \end{aligned}$$

where e_m denotes the final projection of s_m.

Now the problem is the product $u^r e_m u^{-r} u^s e_n u^{-s}$ may not be of the form $u^k e_c u^{-k}$ for some k and c. Nevertheless it will be in the linear span of $\{u^k e_{mn} u^{-k} : k \in R/(mn)\}$. Let P denote the set of projections in $U[R]$ which is in the linear span of $\{u^r e_m u^{-r} : r \in R/(m)\}$ for some m. Explicity, a projection $e \in U[R]$ is in P if and only if there exists an $m \in R^\times$ and $a_r \in \{0,1\}$ such that $e = \sum_r a_r u^r e_m u^{-r}$.

Now it is easy to show that P is a commutative semigroup of projections containing 0. Moreover P is invariant under conjugation by u^r, s_m and s_m^*. One can prove the following Proposition just as in the case when $R = \mathbb{Z}$.

Proposition 8.2. *Let $T := \{s_m^* u^n e u^{n'} s_{m'} : e \in P, m, m' \neq 0, n, n' \in R\}$. Then T is an inverse semigroup of partial isometries. Moreover the set of projections in T coincide with P. Also the linear span of T is dense in $U[R]$.*

Let $\hat{R} := \{(r_m) \in \prod R/(m) : r_{mk} = r_m \text{ in } R/(m)\}$ be the profinite completion of the ring R. For $r \in \hat{R}$, define

$$A_r := \{f \in P : f \geq u^{r_m} e_m u^{-r_m} \text{ for some } m\}$$

Then A_r is an ultrafilter for every $r \in \hat{R}$ and the map $r \to A_r$ is a topological isomorphism from $\hat{R}$ to $\hat{P}_{tight}$.

Let $Q(R)$ be the field of fractions of R. For $m \neq 0$, let $\mathcal{R}_m := \hat{R}$. For every $\ell \neq 0$, let $\phi_{m\ell,m} : \mathcal{R}_m \to \mathcal{R}_{\ell m}$ be the map defined by multiplication by ℓ. Then $\phi_{m\ell,m}$ is only an additive homomorphism and it does not preserve the multiplication. We let $\mathcal{R}$ be the inductive limit of $(\mathcal{R}_m, \phi_{m\ell,m})$. Then $\mathcal{R}$ is an abelian group and $\hat{R}$ is a subgroup of $\mathcal{R}$ via the inclusion $\mathcal{R}_1 \subset \mathcal{R}$. Note that $\mathcal{R}$ is a locally compact Hausdorff space. Moreover the group

$$P_{Q(R)} := \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in Q(R)^\times, b \in Q(R) \right\}$$

acts on $\mathcal{R}$ by affine transformations. The action is descibed explicitly by the following formula. For $x \in \mathcal{R}_p$

$$\begin{bmatrix} 1 & 0 \\ \frac{n}{m'} & \frac{m}{m'} \end{bmatrix} x = mx + np \in \mathcal{R}_{m'p}$$

One can check that the above formula defines an action of $P_{Q(R)}$ on $\mathcal{R}$. Let $\mathcal{G}_{tight}$ be the tight groupoid associated to the inverse semigroup T defined in Proposition 8.2. Then as in the case when $R = \mathbb{Z}$, we have the following theorem.

Theorem 8.3. *Let $\phi : \mathcal{R} \times P_{Q(R)}|_{\hat{R}} \to \mathcal{G}_{tight}$ be the map defined by*

$$\phi\Big((r, \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix})\Big) = [(r, s_m^* u^k s_n)]$$

Then ϕ is a topological groupoid isomorphism. Moreover the C^*-algebra $U[R]$ is isomorphic to the full (and the reduced) C^*-algebra of the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$.

We end this article by showing that $U[R]$ is simple.

Proposition 8.4 ([CL10])**.** *The C^*-algebra $U[R]$ is simple.*

Proof. Let us denote the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$ by $\mathcal{G}$. As in Proposition 6.1, we need to show that $\mathcal{G}$ is minimal and topologically principal. The proof of the minimality of $\mathcal{G}$ is exactly similar to that in Proposition 6.1. We now show that $\mathcal{G}$ is topologically principal. For $g \in P_{Q(R)} \backslash \{1\}$, let us denote the set of fixed points of g in $\hat{R}$ by F_g. It follows from Baire category theorem that $\mathcal{G}$ is topologically principal if and only if F_g has empty interior for every $g \neq 1$.

Let $g = \begin{bmatrix} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{bmatrix}$ be a non-identity element in $P_{Q(R)}$. Suppose that F_g contains a non-empty open set say U. Now note that R is dense in $\hat{R}$. Thus $U \cap R$ is non-empty. Moreover $U \cap R$ is infinite. Let r_1, r_2 be two distinct points of R in U. Since $r_1, r_2 \in F_g$, it follows that $mr_1 - k = nr_1$ and $mr_2 - k = nr_2$. Thus we have $(m-n)r_1 = k = (m-n)r_2$. This forces $m = n$ and $k = 0$. This is a contradiction to the fact that $g \neq 1$. Thus for every $g \neq 1$, F_g has empty interior which in turn implies that $\mathcal{G}$ is topologically principal. This completes the proof. □

Remark 8.5. *In* [KLQ10]*, Cuntz-Li type relations arising out of a semidirect product $N \rtimes H$ where N is a normal subgroup and H is an abelian group satisfying certain hypothesis were considered. It was shown in* [KLQ10] *that the universal C^*-algebra generated by the Cuntz-Li type relations is isomorphic to a corner of a crossed product algebra. It is possible to apply inverse semigroups and tight representations to reconstruct this result. The details will be spelt out elsewhere.*

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