ON TESTS FOR SOME STATISTICAL HYPOTHESES FOR DEPENDENT COMPETING RISKS

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Introduction

Competing risks data: $T =$ failure time
$\delta =$ an indicator of the failure type, $\delta \in \{1, 2, \ldots, k\}$

Planning studies: T could be the time to first employment and δ indicating the employment obtained.

Acturial / Medical / engineering studies

Sub-distribution function :

$$F(i, t) = P[T \le t, \delta = i],$$

Sub-survival function :

$$S(i, t) = P[T > t, \delta = i],$$

Sub-density function :$f(i, t)$, $i = 1, 2, \ldots, k$.

The distribution function of T : $H(t) = \sum_{i=1}^{k} F(i, t)$,

the survivor function of T : $S(t) = \sum_{i=1}^{k} S(i, t)$,

the density function of T is $h(t) = \sum_{i=1}^{k} f(i, t)$,

the probability of failure due to the ith risk $p_i = F(i, \infty)$,

cause-specific hazard rate is $\lambda_i(t) = \frac{f(i,t)}{S(t)}$.

The latent failure time model:
$X_1, X_2, \ldots, X_k$ are latent failure times of any unit exposed to k risks.

Observe $T = \min(X_1, X_2, \ldots, X_k)$,

$\delta = j$ if $X_j = \min X_i, 1 \leq i \leq j$.

Under independence the marginal and hence the joint distribution is identifiable from the probability distribution of the observable random variables (T, δ).

The independence or otherwise of the latent lifetimes $(X_1, X_2, \ldots, X_k)$ cannot be statistically tested from any data collected on (T, δ) and has to be assumed on the basis of a priori information, if any.

The marginal distribution functions $F_i(x)$ of X_i may not represent the probability distribution of lifetimes in any practical situation.

Elimination of jth risk may change the environment in such a way that $F_i(x)$ does not represent the lifetime of X_i in the changed scenario.

Locally Most Powerful Rank Tests

Suppose $k = 2$, that is , a unit is exposed to two risks of failure denoted by 1 and 0.

n units are put to trial , the data consists of $(T_i, {\delta^*}_i), i = 1, \ldots, n$ where $\delta^* = 2 - \delta$.

Test the hypothesis $H_0 : F(1, t) = F(2, t),$ for all t.

Under H_0 the two risks are equally effective. However under the alternative hypothesis one would expect that the two risks are not equally effective atleast at some ages.

The likelihood function for n units is given by (see , Aras and Deshpande (1992))

$$L(\underline{T}, \underline{\delta^*}) = \prod_{i=1}^{n} [f(1, t_i)]^{{\delta^*}_i} [f(2, t_i)]^{1-{\delta^*}_i}$$

If $F(i, t)$ depends upon the parameter θ , then inference about it can be based on the above likelihood function.

When T and δ^* are independent then Deshpande (1990) proposed the model

$$F(1,t) = \theta H(t), \;\; F(2,t) = (1-\theta)H(t).$$

$\theta = P[\delta^* = 1]$, and the likelihood reduces to

$$L(\theta, H) = \theta^{\sum_{i=1}^{n} \delta^*{}_i}(1-\theta)^{\sum_{i=1}^{n}(1-\delta^*{}_i)} \prod_{i=1}^{n} h(t_i).$$

The hypothesis $F(1,t) = F(2,t)$ reduces to testing that $\theta = 1/2$.

Then the obvious statistic is the sign statistic $U_1 = \frac{1}{n}\sum_{i=1}^{n} \delta^*{}_i$.

nU_1 has $B(n,\theta)$ distribution and one can have optimal estimation and testing procedures based on it.

However, if $F(1,t)$ and $F(2,t)$ depend on a parameter θ in a more complicated manner then one needs to look at locally most powerful rank tests.

Let $f(1,t) = f(t,\theta), \;\; f(2,t) = h(t) - f(t,\theta)$

$h(t)$ and $f(t,\theta)$ are known density functions and incidence density such that $f(t,\theta_0) = \frac{1}{2}h(t)$.

Let $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(n)}$ denote the ordered failure times.

$$W_i \;=\; \begin{cases} 1 & \text{if } T_{(i)} \text{ corresponds to first risk} \\ 0 & \text{otherwise.} \end{cases} \quad .$$

Let R_j be the rank of T_j among $T_1, \ldots, T_n$.

The likelihood of ranks is given by

$$\int_{0<t_1<\ldots<t_n<\infty} \prod_{i=1}^{n} [f(t_i,\theta)]^{w_i} [h(t_i) - f(t_i,\theta)]^{1-w_i} dt_i$$

Theorem : If $f'(t,\theta)$ is the derivative of $f(t,\theta)$ with respect to θ, then the locally most powerful rank test for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ is given by: reject H_0 for large values of

$L_c = \sum_{i=1}^{n} w_i a_i$, where

$$a_i = \int \ldots \int_{0<t_1<\ldots<t_n<\infty} \frac{f'(t_i,\theta_0)}{f(t_i,\theta_0)} \prod_{i=1}^{n} [f(t_i,\theta_0) dt_i].$$

Special cases

(i) If the independence model holds with $\theta_0 = 1/2$, then sign test is the LMPR test.

(ii) If $f(1,t) = \frac{1}{2}g(t,0)$ and $f(2,t) = \frac{1}{2}g(t,\theta), \theta > 0$ and $g(t,\theta)$ is the logistic density function $g(t,\theta) = \frac{e^{(x-\theta)}}{[1+e^{(x-\theta)}]^2}$, then the LMPR test is based on the statistic

$W^+ = \sum_{i=1}^{n} W_i R_i$.

(iii) In case of Lehmann type alternative defined by $F(1,t) = [\frac{H(t)}{2}]^\theta, \quad F(2,t) = H(t) - [\frac{H(t)}{2}]^\theta$,, LMPR test is based on scores $a_i = E(E_{(j)})$ where $E_{(j)}$ is the jth order statistic from a random sample of size n from standard exponential distribution.

For more complicated families of distributions, e.g., Gumbel (1960), the scores are complicated and need to be solved using numerical integration (see Aras and Deshpande (1992)).

Tests for bivariate symmetry

Suppose the latent failure times X and Y are dependent.

Their joint distribution is given by $F(x, y)$.

On the basis of independent pairs $(T_i, {\delta^*}_i)$ we want to test whether the forces of two risks are equivalent against the alternative that the force of one risk is greater than that of the other.

$H_0 : F(x, y) = F(y, x)$ for every (x, y)

Theorem : Under the null hypothesis of bivariate symmetry we have

(i)$F(1, t) = F(2, t)$ for all t,

(ii)$S(1, t) = S(2, t)$ for all t,

(iii) $\lambda(1, t) = \lambda(2, t)$ for all t,

(iv) $P[\delta^* = 1] = P[\delta^* = 0]$

(v) T and δ^* are independent.

$H_1 : \lambda(1,t) < \lambda(2,t)$

$H_2 : F(1,t) < F(2,t)$

$H_3 : S(1,t) > S(2,t)$.

All these alternatives say that risk II is more potent than risk I at all ages t in some stochastic sense.

Sen (1979) considered fixed sample and sequential tests for the null hypothesis of bivariate symmetry of the joint distribution of (X,Y). The alternatives are expressed in terms of $\pi_1(t) = Pr[\tilde{\delta} = 1|T = t]$,
the conditional probability that the failure is due to first risk , given that failure occurs at time t.

He derived optimal score statistics for such parametric situations. But the statistics cannot be used without the knowledge of the joint distribution $F(x,y)$.

For testing H_0 against H_1 consider

$\psi(t) = F(1,t) - F(2,t)$

$\int_0^t S(t)[\lambda(1,u) - \lambda(2,u)]du.$

H_1 holds iff the above function is non-increasing on t.

Consider the following measure of deviation between H_0 and H_1,

$$\Delta = \int_{0<x<y<\infty} [\psi(x) - \psi(y)]\, dF(x)\, dF(y)$$

Its empirical estimator is Δ_n

$\Delta_n = \int_{0<x<y<\infty} [\psi_n(x) - \psi_n(y)]\, dF_n(x)\, dF_n(y)$

where

$F_{1n}(t) = \frac{1}{n} \sum_{j=1}^{n} I\{\delta_j = 1, T_j \leq t\},$

$F_n(t) = \frac{1}{n} \sum_{j=1}^{n} I\{T_j \leq t\},$

$\psi_n(t) = 2F_{1n}(t) - F_n(t),$

are the empirical estimators of F_1, F and ψ, respectively.

Then we can write

$\Delta_n = \frac{1}{n^3}[\frac{n(n^2-1)}{6} - 2\sum_{i=1}^{n}(i-1)(n-i+1)W_i]$.

Under H_0, $\Delta = 0$, under H_1, $\Delta > 0$.

Large values of the statistic are significant. Rejecting H_0 for large values of Δ_n is equivalent to rejecting it for small values of the statistic

$U_2 = \sum_{i=1}^{n-1} i(n-i)W_{i+1}$.

T and δ are independent under H_0.

Therefore $W_1, W_2, \ldots, W_n$ are i.i.d. Bernoulli random variables with

$P\{W_i = 1\} = P\{W_i = 0\} = \frac{1}{2}$, $i = 1, 2, \ldots, n$.

The moment generating function of the null distribution of U_2 is given by

$M(t) = 2^{-n+1}\prod_{i=1}^{n-1}(1 + e^{a_i t})$,

where $a_i = i(n-i), i = 1, 2, \cdots, n-1$.

For $n = 5(1)20$, the 5% and 1% critical values of U_2 are given below.

Critical points of U_2 and exact significance levels.

n	$\alpha \approx 0.01$	$\alpha \approx 0.05$
5	-	0, 0.0625
6	-	0, 0.03125
7	0, 0.015625	6, 0.046875
8	7, 0.015625	14, 0.046875
9	8, 0.011719	22, 0.054688
10	18, 0.011719	34, 0.054688
11	28, 0.013672	50, 0.050781
12	38, 0.010254	67, 0.051758
13	54, 0.010498	90, 0.051514
14	74, 0.01001	118, 0.052734
15	98, 0.010132	148, 0.050293
16	126, 0.01001	186, 0.0513
17	158, 0.01001	228, 0.050308
18	198, 0.010269	278, 0.050613
19	242, 0.010311	332, 0.049911
20	290, 0.01005	396, 0.050467

$$E[U_2] = \frac{n(n^2-1)}{12}, \quad Var[U_2] = \frac{n(n^4-1)}{120}$$

By the Central Limit Theorem, it can be shown that, under H_0,

$$n^{\frac{1}{2}}\{\frac{U_2}{n^3} - \frac{1}{12}\} \xrightarrow{\mathcal{L}} N(0, \frac{1}{120}).$$

To give an idea about the accuracy of the normal approximation of U_2, we give the exact (respectively, approximate) significance levels for $n = 10$ and $n = 20$.

For $n = 10$, $P[U_2 \leq 18] = 0.011719$ (resp. 0.0118) and $P[U_2 \leq 34] = 0.054688$ (resp. 0.0594).

For $n = 20$, $P[U_2 \leq 290]= 0.01005$ (resp. 0.0111) and $P[U_2 \leq 396] = 0.050467$ (resp. 0.049).

Deshpande (1990) proposed two tests for testing H_0 versus $H_2 : F(1,t) < F(2,t)$ on heuristic grounds.

The first test is the Wilcoxon signed rank type statistic

$$W^+ = \sum_{i=1}^{n} (1 - \delta^*{}_i) R_i.$$

It was felt that W^+ will be large when the alternative H_2 is true, there being a greater incidence of the second risk upto any fixed time t.

Another test is based on the U-statistic

$$U_3 = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_3(T_i, \delta^*{}_i, T_j, \delta^*{}_j)$$

$$\phi_3(T_i, \delta^*{}_i, T_j, \delta^*{}_j) = \begin{cases} 1 & \text{if } \delta^*{}_j = 0, T_i > T_j \\ & \text{or } \ \delta^*{}_i = 0, T_i < T_j \\ 0 & \text{otherwise} \end{cases}$$

The kernel ϕ_3 takes value 1 if , and only if, a Y observation is the smallest among (X_i, X_j, Y_i, Y_j).

$\binom{n}{2} U_3 = \sum_{i=1}^{n} (n - R_i + 1)\delta^*{}_i$.

$E(U_3) = 1/2$ under H_0 and strictly larger than 1/2 under H_2.

U_3 is same as the statistic proposed earlier to test for H_0 against the alternative H_{A1} of stochastic dominance of distribution functions of independent latent failure times (see, Deshpande and Dewan (2003)) is also consistent for testing bivariate symmetry against dominance of incidence functions.

For testing H_0 against H_2, one can consider the measure of deviation $F(2,t)-F(1,t)$, which is non-negative under H_2.

$$\int_0^\infty [F(2,t) - F(1,t)]dH(t)$$
$$= P[\delta^*{}_1 = 0, T_1 \leq T_2] - \frac{1}{2}.$$

A U-statistic estimator of this parameter is the statistic U_3 discussed above.

Similarly for testing H_0 against H_3 consider the measure of deviation $S(1,t) - S(2,t)$, which is non-negative under H_3. Then

$$\int_0^\infty [S(1,t) - S(2,t)]dH(t)$$
$$= P[\delta^*{}_1 = 1, T_1 > T_2] - \frac{1}{2}.$$

Consider the kernel

$$\phi_4(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } \delta^*{}_i = 1, T_i > T_j \\ & \text{or } \delta^*{}_j = 1, T_i < T_j \\ 0 & \text{otherwise} \end{cases}$$

The corresponding U-statistic is given by

$$U_4 = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \phi_4(T_i, \delta_i, T_j, \delta_j).$$

$\binom{n}{2}U_4 = \sum_{i=1}^{n}(R_i - 1)\delta^{*}{}_i.$

This statistic was earlier proposed by Bagai, Deshpande and Kochar (1989) to test for equality of failure rates of independent latent competing risks.

Aly , Kochar and Mckeague (1994) proposed Kolmogrov-Smirnov type tests for testing the equality of two competing risks against the alternatives H_1 and H_2.

$$\begin{aligned}\psi^*(t) &= F(2,t) - F(1,t) \\ &= \int_0^t S(u)(\lambda(2,u) - \lambda(1,u))du.\end{aligned}$$

Under H_0, $\psi^*(t) = 0$.

H_1 holds iff $\psi^*(t)$ is nondecreasing in t .

Let $\psi_n^*(t)$ be its empirical estimator.

$D_{1n} = \sup_{0\le s<t<\infty}\{\psi_n^*(t) - \psi_n^*(s)\}.$

Large values of the statistic are significant.

Exact null distribution of D_{1n} is given below for $t = 1, \ldots, n+1$,

$$P[nD_{1n} < t]$$

$$= \frac{2}{2t+1} \sum_{j=0}^{2t} (\cos \frac{j\pi}{2t+1})(\sin \frac{j\pi(t+1)}{2t+1})$$

$$\times (1 + \cos \frac{j\pi}{2t+1})(\frac{1-(-1)^j}{2}) / \sin \frac{j\pi}{2t+1}.$$

Asymptotic null distribution of D_{1n} is given by

$$\sqrt{n} D_{1n} \xrightarrow{\mathcal{L}} \sup_{0 \leq x \leq 1} |W(x)|,$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion.

The asymptotic $0.90, 0.95, 0.99$ quantiles of $\sqrt{n} D_{1n}$ were found to be $1.96, 2.241, 2, 807$, respectively.

For a general two sided alternative $F(1,t) \neq F(2,t)$ for some t or equivalently $\lambda_1(t) \neq \lambda_2(t)$ for some t , then one can use the Kolmogrov-Smirnov type statistic

$D_n = \sup_{t \geq 0} |\psi_n^*(t)|$.

Under $H_0, \sqrt{n} D_n \xrightarrow{\mathcal{L}} \sup_{0 \leq x \leq 1} |W(x)|$.

This test is consistent against arbitrary departures from H_0.

For testing H_0 against H_2, Aly , Kochar and McKeague (1994) proposed the statistic

$D_{2n} = \sup_{0 \leq t \infty} \psi_n(t)$.

Large values of D_{2n} are significant for testing H_0 against H_2.

The exact nulltribution of D_{2n} is

$P[nD_{2n} = k] = \frac{1}{2^n}\binom{n}{[\frac{n-k}{2}]}, \;\; k = 0, 1, \ldots, n.$

The asymptotic null distribution is

$P[\sqrt{n} D_{2n} > x] \to 2(1 - \Phi(x)), \;\; x \geq 0,$

where Φ is the standard normal distribution function.

Remarks

(i) The various tests are consistent against their intended alternatives.

(ii) The tests are distribution-free under H_0. The null distribution of the tests U_3 and U_4 is same as in the case of independent latent failures (see, Deshpande and Dewan (2003)).

(iii) We can also use these tests for the hypothesis $\lambda_1(t) = \lambda_2(t)$ against the alternative that cause-specific hazards are ordered.

(iv) T and δ continue to be independent under the null hypothesis of bivariate symmetry .

(v) The statistic U_2 puts more weight on the middle observations and is less sensitive to the observations in the beginning and the end of the experiment.

U_3 puts more weight to later observations and

U_4 puts higher weight to observations in the beginning.

(vi) The statistics U_2, U_3, U_4 are all linear combinations of the sign statistic and the Wilcoxon-signed rank type statistic.

(vii) Tests proposed by Aly, Kochar and Mckeague (1994) can be extended to the case of multiple risks in which any two of the cause-specific risks are to be compared. The statistic can be modified to test dominance of one risk over the other in a specified interval.

Censored data

Let C be the censoring random variable independent of the latent failure times X and Y.

The survival function of C is S_c and assume that $S_C(t) > 0$ for all t.

The available information consists of $(\tilde{T}_i, \tilde{\delta}_i)$, $i = 1, 2, \cdots, n$

$\tilde{T} = \min(T, C)$ and $\tilde{\delta} = \delta^* I(T \leq C)$.

Aly et al (1994) generalised the function ψ^* so as to capture departures of H_0 from H_2 in case of censored data.

$$\phi(t) = \int_0^t S(u-)(S_C(u-))^{\frac{1}{2}}(\lambda(2,u) - \lambda(1,u))du$$

which is the ψ^* function when there is no censoring.

The integrand $S_C(u-)^{1/2}$ is the function required to compensate for censoring in order that the D statistics remain asymptotically distributio free.

Under $H_0, \phi(t) = 0$.

H_1 holds iff $\phi(t)$ is increasing in t.

The relevant statistic is

$$D_{3n} = \sup_{0 \leq s < t < \infty}\{\phi_n(t) - \phi_n(s)\}.$$

An obvious choice of ϕ_n is

$$\phi_n(t) = \int_0^t \widehat{S}(u-)(\widehat{S}_C(u-))^{1/2}\, d(\widehat{\Lambda}_1 - \widehat{\Lambda}_2)(u)$$

$\widehat{S}$ and $\widehat{S}_C$ are the product limit estimators of $\bar{S}$ and S_C,

$\widehat{\Lambda}_j$ is the Aalen estimator of the cumulative CSHR function $\Lambda_j(t) = \int_0^t \lambda(j,u)\,du$.

$\widehat{\Lambda}(j,t) = \sum_{i|\tilde{T}_i \le t} \frac{I(\tilde{\delta}_i = j)}{R_i}$

where $R_i = \#\{k : \tilde{T}_k \ge \tilde{T}_i\}$ is the risk set at time $\tilde{T}_i$.

Large values of the statistic are significant.

Since $\phi(t) > 0$ for some t under H_2, a suitable test procedure is based on large values of

$D_{4n} = \sup_{0 \le t < \infty} \phi_n(t)$.

Aly et al (1994) showed that D_{3n} and D_{4n} are asymptotically distribution-free with the same limiting distributions as those obtained in the uncensored case.

A suitable modification of U_2 to censored data is given by the statistic

$K_n = \int_{0<x<y<\infty} (\phi_n(x) - \phi_n(y))\, d\widehat{S}(x)\, d\widehat{S}(y)$

where ϕ_n and $\widehat{S}$ are as defined above.

K_n can be expressed as

$K_n = \sum_{i|\tilde{\delta}_i \neq 0} \phi_n(i)(2\widehat{S}(\tilde{T}_{(i-1)}) - 1)(\widehat{S}(\tilde{T}_{(i-1)})\frac{1}{n-i+1}$

Large values of K_n are significant for testing H_0 against H_1.

Aly et al (1994) showed that, under H_0,

$n^{1/2}\phi_n \quad \xrightarrow{\mathcal{L}} \quad W(S(.))$

where W(.) is the standard Brownian Motion.

Using the continuous mapping theorem, we have

$n^{1/2}K_n \xrightarrow{\mathcal{L}} N(0, \sigma^2)$

$\sigma^2 = \frac{1}{30}$

Sun and Tiwari (1995) modified the statistic U_3 so that it can be used for censored data.

$$\begin{aligned} V &= \int_0^\infty (F(2,t) - F(1,t))dH(t) \\ &= \int_0^\infty [S(t)]^2 d(\Lambda_2(t) - \Lambda_1(t)) \end{aligned}$$

A natural estimator of V is given by V_n where

$V_n = \int_0^\infty [\widehat{S}(t-)]^2 d(\widehat{\Lambda}_2(t) - \widehat{\Lambda}_1(t))$

where $\widehat{S}(t)$ and $\widehat{\Lambda}_j$ are as defined above.

In the absence of censoring V_n reduces to the statistic U_3.

Then Sun and Tiwari (1995) proved the following theorem.

Theorem: Let $K(t) = 1 - S(t)S_C(t)$, $\tau_K = \sup\{t : K(t) < 1\}$. If $\int_0^{\tau_K} \frac{dH(t)}{S_C(t)} < \infty$, then

$\sqrt{n}(V_n - V) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \ \ as n \to \infty.$

Under H_0,

$\sigma^2 = \int_0^\infty \frac{S^2(t)}{S_C(t)} dH(t)$.

A consistent estimator $\hat{\sigma}^2$ of σ^2 is obtained by replacing S ans S_C by their consistent Kaplan Meier-estimators.

When there is no censoring $\sigma^2 = \frac{1}{3}$.

The test rejects for large values of the statistic.

The approximate power of this test of size α is equal to $1 - \Phi(z_\alpha - \sqrt{n}V/\hat{\sigma})$, where z_α is the upper α percentile of standard normal distribution.

SIMULATION RESULTS

Random samples were generated from absolutely continuous bivariate exponential (ACBVE) due to Block and Basu (1974) with density

$$f(x,y) \;=\; \begin{cases} \frac{\lambda\lambda_1(\lambda_2+\lambda_0)}{\lambda_1+\lambda_2} e^{-\lambda_1 x-(\lambda_2+\lambda_0)y} & \text{if} \quad x<y, \\ \frac{\lambda\lambda_2(\lambda_1+\lambda_0)}{\lambda_1+\lambda_2} e^{-\lambda_2 y-(\lambda_1+\lambda_0)x} & \text{if} \quad x>y, \end{cases}$$

where $(\lambda_0, \lambda_1, \lambda_2)$ are the parameters and $\lambda = \lambda_0 + \lambda_1 + \lambda_2$. The CSHR's are

$$\lambda_j(t) = \frac{\lambda_j \lambda}{\lambda_1 + \lambda_2}, \quad j = 1, 2.$$

Under $H_1 \quad \lambda_1 < \lambda_2$.

X and Y are independent if and only if $\lambda_0 = 0$.

We set $\lambda_1 = 1$ and consider $\lambda_2 = 1.0, 1.4, 1.8, 2.2$ indicating larger and larger departures from H_0.

The case $\lambda_2 = 1.0$ corresponds to the null hypothesis. $n = 100$ and there are 10000 replications.

	λ_2			
Test	1.0	1.4	1.8	2.2
D_1	3.76	41.98	82.53	96.83
D_2	4.85	47.71	86.98	98.14
U_2	5.09	44.60	83.96	96.92
U_3	4.79	43.06	80.54	95.42
U_4	4.99	42.67	80.81	95.77
$Sign$	4.39	49.50	88.29	98.66

Next we look at the censored case. The censoring distribution was exponential with parameters 1 and 3, respectively. We use asymptotic critical levels of 5 percent. Results are based on $5,000$ replications.

Observed levels and powers of K_n at an asymptotic level of 5 percent. The underlying distribution of (X, Y) is Block and Basu (1974) ACBVE with $\lambda_1 = 1$.

(a) CENSORED (EXP(1))

	n=50		n=100	
λ_2	$\lambda_0 = 0$	$\lambda_0 = 1$	$\lambda_0 = 0$	$\lambda_0 = 1$
1.0	.0218	.0312	.0360	.0376
1.5	.1864	.2192	.3732	.4302
2.0	.4482	.5080	.7862	.8342
2.5	.6928	.7414	.9546	.9704

(b) CENSORED (EXP(3))

	n=50		n=100	
λ_2	$\lambda_0 = 0$	$\lambda_0 = 1$	$\lambda_0 = 0$	$\lambda_0 = 1$
1.0	.0048	.0124	.0084	.0172
1.5	.0342	.0762	.1012	.1834
2.0	.1202	.1986	.3216	.4860
2.5	.2496	.3774	.5862	.7344

From the table it is clear that the asymptotic critical levels give conservative tests for the censored case, with the effect increasing as the censoring becomes more severe.

There is slight effect on the levels or the power due to lack of independence of X and Y in the presence of censoring.

The results are comparable with the test proposed by Aly et al (1994) for the lightly censored case.

Test for independence of T and δ

The nature of dependence between T and δ is crucial and useful in modelling competing risks data via sub-distribution/subsurvival functions.

If T and δ are independent then $S_i(t) = pr(\delta = i)S(t)$, allowing the study of the failure times and the causes (risks) of failure separately.

The hypothesis of equality of incidence functions or that of cause-specific hazard rates reduces to testing whether $pr(\delta = 1) = pr(\delta = 0) = 1/2$.

This simplifies the study of competing risks to a great extent.

Dewan, Deshpande and Kulathinal (2004) studied the properties of the conditional probability functions
$\Phi_i(t) = pr(\delta = i \mid T \geq t) = S_i(t)/S(t),\ i = 1, 2$
$\Phi_i^*(t) = pr(\delta = i \mid T < t) = F_i(t)/H(t),\ i = 1, 2$.

(i) T and δ are independent iff
$\Phi_1(t) = P[\delta = 1]$ or $\Phi_2^*(t) = 1 - P[\delta = 1]$

(ii) T and δ are PQD iff
$\Phi_1(t) \geq P[\delta = 1]$ or $\Phi_2^*(t) \geq 1 - P[\delta = 1]$

(iii) δ is Right Tail Increasing in T iff $\Phi_1(t)$ is increasing in t.

(iv) δ is Left Tail Decreasing in T iff $\Phi_2^*(t)$ is decreasing in t.

They considered the problem of testing H_0 : T and δ are independent which is equivalent to

$$H_0 : \Phi_1(t) \text{ is a constant}$$

against various alternative hypotheses which characterise the properties of $\Phi_1(t)$ and $\Phi_0^*(t)$:

$H_1 : \Phi_1(t)$ is not a constant

$H_2 : \Phi_1(t) \geq P[\delta = 1]$ for all t with strict inequality f

$H_3 : \Phi_1(t)$ is a monotone nondecreasing function of

$H_4 : \Phi_0^*(t)$ is a monotone nonincreasing function of

If the hypothesis of independence is accepted then one can simplify the model and study the failure time and cause of failure separately.

If the hypothesis is rejected then a suitable model under specific dependence between T and δ in terms of the incidence functions is needed.

The tests constructed for the two risk case cannot be straightway extended to the case of more than 2 causes of failure.

For example, in the most commonly cited mortality data given in Hoel (1972), the data were obtained from a laboratory experiment on two groups of RFM strain male mice which had received a radiation dose of 300 r at an age of 5-6 weeks.

The first group of mice lived in a conventional laboratory environment, while the second group was in a germ-free environment.

The causes of death were grouped into three classes - thymic lymphoma (risk=1), reticulum cell sarcoma (risk=2) and all other causes (risk=3).

Suppose we have information on (T, δ), where δ takes three values $1, 2, 3$.

Let $\phi_i(t) = \frac{S_i(t)}{S(t)}$.

(T, δ) are PQD iff

$\phi_3(t) \geq \phi_3(0) = \phi_3$,

and

$\phi_1(t) \leq \phi_1(0) = \phi_1$,

Similarly δ is RTI in T iff

$\phi_3(t)$ is increasing in t

and

$\phi_1(t)$ is decreasing in t.

Tests For PQD

$\int_0^\infty [S_3(t) - \phi_3 S(t)]dF(t) = P[T_2 > T_1, \delta_2 = 3]$

and

$\int_0^\infty [S_1(t) - \phi_1 S(t)]dF(t) = P[T_2 > T_1, \delta_2 = 1]$.

Consider

$$\psi_1(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1 & \text{if } T_j > T_i, \delta_i = 1, \delta_j = 3, \\ & T_j > T_i, \delta_i = 2, \delta_j = 3, \\ & T_j > T_i, \delta_i = 3, \delta_j = 3, \\ & \text{or if } T_i > T_j, \delta_i = 3, \delta_j = 1, \\ & T_i > T_j, \delta_i = 3, \delta_j = 2, \\ & T_i > T_j, \delta_i = 3, \delta_j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let U_1 be the corresponding U-statistic.

Further consider

$$\psi_2(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1 & \text{if } T_j > T_i, \delta_i = 1, \delta_j = 1, \\ & T_j > T_i, \delta_i = 2, \delta_j = 1, \\ & T_j > T_i, \delta_i = 3, \delta_j = 1, \\ & \text{or if } T_i > T_j, \delta_i = 1, \delta_j = 1, \\ & T_i > T_j, \delta_i = 1, \delta_j = 2, \\ & T_i > T_j, \delta_i = 1, \delta_j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let U_2 be the corresponding U-statistic.

Then $U_1 - U_2$ can be used to test for PQD.

Large values of the statistic are significant.

$E(U_1) = 2\int_0^\infty S_3(t)dF(t), \;\; E(U_2) = 2\int_0^\infty S_1(t)dF(t).$

In particular under the null hypotheses

$E(U_1) = \phi_3, \;\; E(U_2) = \phi_1.$

Under H_0,

$\sigma_1^2 = \frac{1}{3}\phi_3(1 - \phi_3),$

$\sigma_2^2 = \frac{1}{3}\phi_1(1 - \phi_1).$

Theorem : $\sqrt{n}(U_i - E(U_i)) \xrightarrow{\mathcal{L}} \;\; N(0, \sigma_i^2), \;\; i = 1, 2$ as $n \to \infty, i = 1, 2.$

Under H_0

$Cov(U_1, U_2) = -\frac{2\phi_1\phi_3}{3}\frac{2n-1}{n(n-1)}.$

Hence , under H_0 , limiting variance of $\sqrt{n}(U_1 - U_2)$ is given by

$\sigma^2 = \frac{4}{3}(\phi_1 + \phi_3) - \frac{4}{3}(\phi_1 - \phi_3)^2.$

And $\sqrt{n}(U_1 - U_2) \xrightarrow{\mathcal{L}} N(0, \sigma^2), \quad$ as $n \to \infty$.

One can use asymptotic critical points for testing purposes.

One can easily find the exact distribution of U_1, U_2.

Even the tests for independence of T and δ are linear combinations of the sign statistic and the signed rank statistic .

MODELS TO FIT THIS SET UP?