ESTIMATING PARAMETERS IN MULTICHANNEL SINUSOIDAL MODEL

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Abstract. In this paper, we study the problem of estimation of parameters of multichannel sinusoidal model. In multichannel sinusoidal model, the inherent frequencies from distinct channels are same with different amplitudes. It is assumed that the errors in individual channel are independently and identically distributed whereas the signal from different channels are correlated. We first propose to minimize the sum of residual sum of squares to estimate the unknown parameters, and they can be easily obtained. Next we propose to use more efficient generalized least squares estimators and which become the maximum likelihood estimators also when the errors follow multivariate Gaussian distribution. Both the estimators are strongly consistent and asymptotically normally distributed. We have provided the implementation of the generalized least squares estimators. Simulation experiments have been performed to compare the performances of the least squares estimators and generalized least squares estimators. It is observed that the variances of the maximum likelihood estimators reach the Cramer-Rao lower bound even for moderate sample sizes. We have extended the methods of estimation and the associated results of the two-channel model to an arbitrary m-channel model. It is observed that the computational complexity does not increase significantly with the increase of number of channels.

1. Introduction

The problem of finding sinusoidal parameters received at multichannel outputs has several applications such as particle size and velocity estimation in laser anemometry, Handel and Host-Madsen [5], impedance measurement, Ramos, da Silva and Serra [8], electric power calibration, Vucijak and Saranovac [13] etc. The problem has a long history starting with the work of Sakai [10]. In recent time also it has received a considerable amount of attention in the Signal Processing literature, see for example Clercq et al. [2], Sandgren et al. [11], Papy, Lathauwer and Van Huffel [7], Handel [4], Griffin et al. [3], So and Zhou [14], Chan, So and Sun [1], Zhou, So and Christensen [17], Stanović et al. [12] and the references cited therein. Zhou [15] studied the spectral analysis of multichannel sinusoidal model and recently Zhou

2000 *Mathematics Subject Classification.* 62J02; 62E20; 62C05.

Key words and phrases. Sinusoidal model; Least squares estimator; Generalized least squares estimators; Cramer-Rao lower bound; consistency; asymptotic normality.

et. al [16] discussed a robust method of estimation. In all these cases the authors developed efficient estimators of the unknown parameters based on independent error assumptions of different channels.

In this paper, we consider the estimation of unknown parameters of a multichannel sinusoidal signal model. We have first discussed about two channel model, and then we have generalized it to m-channel model. The two-channel sinusoidal model is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} A_1^0 & B_1^0 \\ A_2^0 & B_2^0 \end{pmatrix} \begin{pmatrix} \cos(\omega^0 t) \\ \sin(\omega^0 t) \end{pmatrix} + \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}. \tag{1}$$

In matrix notation, model (1) can be written as

$$\mathbf{y}(t) = \mathbf{A}^0 \boldsymbol{\theta}(\omega^0, t) + \mathbf{e}(t) \tag{2}$$

$$= \boldsymbol{\mu}(t; \boldsymbol{\beta}^0, \omega^0) + \mathbf{e}(t), \tag{3}$$

where $\mathbf{y}(t) = (y_1(t), y_2(t))^T$, $\mathbf{A}^0 = \begin{pmatrix} A_1^0 & B_1^0 \\ A_2^0 & B_2^0 \end{pmatrix}$, $\boldsymbol{\theta}(\omega^0, t) = (\cos(\omega^0 t), \sin(\omega^0 t))^T$, $\mathbf{e}(t) = (e_1(t), e_2(t))^T$, $\boldsymbol{\beta}^0 = (A_1^0, B_1^0, A_2^0, B_2^0)$ and $\boldsymbol{\mu}(t; \boldsymbol{\beta}^0, \omega^0) = (\mu_1(t; \boldsymbol{\beta}^0, \omega^0), \mu_2(t; \boldsymbol{\beta}^0, \omega^0))^T$. The signal from the k-th channel, $k = 1, 2$, takes the following form;

$$y_k(t) = \Big[A_k^0 \cos(\omega^0 t) + B_k^0 \sin(\omega^0 t)\Big] + e_k(t), \quad t = 1, \ldots, n.$$

Here the bivariate random vector $\mathbf{y}(t)$ represents the signal from the two channels at the time point t; $\omega^0 \in (0, \pi)$ is the common frequency; A_1^0 and B_1^0 are amplitudes corresponding to ω^0 from the first channel and A_2^0 and B_2^0 are from the second channel; the bivariate random vector $\mathbf{e}(t)$ represents the noise part and its explicit structure is stated in the following assumption.

Assumption 1. *The bivariate random vectors $\{\mathbf{e}(t), t = 1, \ldots, n\}$ are independent and identically distributed (i.i.d.) with mean vector $\mathbf{0}$ and dispersion matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$, that is,*

$$\mathbf{e}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \overset{i.i.d.}{\sim} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right] \tag{4}$$

where σ_1^2 and $\sigma_2^2 > 0$.

In the class of multichannel sinusoidal model, two-channel model is the basic model. The aim will be to estimate the unknown frequencies and the elements of matrix $\mathbf{A}^0$ given a sample of size n. One can consider the observations from different channels and apply sinusoidal model separately to estimate the unknown parameters. Then the information that the frequencies are same through different channels is ignored and there will be loss of precision in estimation of the frequencies. Therefore, we need estimation method which estimates the unknown parameters, that is, the frequencies and amplitude parameters of all the channels simultaneously.

In this paper first we provide a very simple estimation procedure of the unknown parameters based on least squares (LS) method. The two-channel model has five unknown parameters and these simple estimators can be obtained by solving a one-dimensional optimization problem. Although, they are not the most efficient estimators, they are consistent and asymptotically normally distributed. To provide efficient estimators, we propose to use generalized least squares estimators (GLSEs). First it is assumed that the noise dispersion matrix $\boldsymbol{\Sigma}$ is known and it is shown that the proposed GLSEs are consistent and they have lower asymptotic variances than the simple least squares estimators (LSEs). Since the noise dispersion matrix $\boldsymbol{\Sigma}$ is usually unknown, we have indicated how the generalized LSEs can be implemented in practice. Simulation experiments have been performed to compare the performances of these estimators and it is observed the the proposed generalized LSEs perform better than the ordinary LSEs in all the cases considered.

It may be mentioned that when the number of channel is one, then the model can be written as

$$y(t) = A^0 \cos(\omega t) + B^0 \sin(\omega t) + e(t), \tag{5}$$

here A^0 and B^0 are the amplitudes, ω is the frequency and $e(t)$s are the additive error with mean zero and variance σ^2. An extensive amount of work has been done in the Statistical Signal Processing literature in developing different estimation procedures of the unknown parameters of the sinusoidal model (5), and developing their properties, see for example the recent monograph by Nandi and Kundu [6] and see the references cited therein in this respect.

The major contribution of this article is the following: This is the first time it has been considered the correlated error in m-channel set-up. In all the existing literature it has been assumed that the noise components in different channels are independently distributed. First we have provided a very simple LSEs of the amplitudes and the frequency, and established the consistency and asymptotic normality properties of these estimators. They are not efficient, but they have been used quite effectively to compute efficient GLSEs when the noise variance-covariance matrix is not known. Moreover, the theoretical properties of the GLSEs have been established and it has been shown that it reaches the Cramer-Rao lower bound under the assumption of multivariate normal distribution. Hence, it can be considered as a benchmark of any estimation procedure. Further, it may be mentioned that it has been observed that the computational complexity does not significantly increase as m increases. It is a significant achievement in terms of theoretical development as well as for implementation purposes.

The rest of the paper is organized as follows. In Section 2, first we consider the simple least squares estimators of the two channel model and provide their theoretical properties. The generalized LSEs, their properties and implementation for two-channel models are discussed in Section 3. In Section 4, we provide the results of the simulation experiments and in Section 5, we present the general case of multichannel sinusoidal model. Finally, we conclude the paper in Section 6.

2. Least Squares Estimators

In this section, we consider the LSEs of the unknown parameters of model (1). Our problem is to estimate the unknown parameters given a sample of size n, $\{\mathbf{y}(1), \ldots, \mathbf{y}(n)\}$. The main aim is to estimate first the elements of the matrix $\mathbf{A}^0$ and ω^0 and then using these estimates, the parameters of the noise process $\{\mathbf{e}(t)\}$ need to be estimated. The consistency and the asymptotic normality of the estimators have been established. It is observed that the estimators of the noise component can be used in implementing the generalized LSEs quite efficiently.

Apart from Assumption 1, we consider the following assumption on the true values of the unknown parameters.

Assumption 2. *For $k = 1, 2$, A_k^0 and B_k^0 are not simultaneously equal to zero.*

In this section, we consider the simple LSEs which can be obtained by minimizing the residual sum of squares from the two channels. Let $\boldsymbol{\xi} = (A_1, B_1, A_2, B_2, \omega)^T$ and $\boldsymbol{\xi}^0$ denote the true value of $\boldsymbol{\xi}$. The LSE $\widetilde{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ in this case minimizes the residual sum of squares with respect to $\boldsymbol{\xi}$, defined as follows:

$$
\begin{aligned}
R(\boldsymbol{\xi}) &= \sum_{t=1}^{n} \mathbf{e}^T(t)\mathbf{e}(t) = \sum_{t=1}^{n}[e_1^2(t) + e_2^2(t)] \\
&= \sum_{t=1}^{n}[y_1(t) - A_1\cos(\omega t) - B_1\sin(\omega t)]^2 + \sum_{t=1}^{n}[y_2(t) - A_1\cos(\omega t) - B_1\sin(\omega t)]^2.
\end{aligned}
$$

Using matrix notation $R(\boldsymbol{\xi})$ can be written as follows:

$$R(\boldsymbol{\xi}) = (\mathbf{Y}_1 - \mathbf{X}(\omega)\boldsymbol{\delta}_1)^T(\mathbf{Y}_1 - \mathbf{X}(\omega)\boldsymbol{\delta}_1) + (\mathbf{Y}_2 - \mathbf{X}(\omega)\boldsymbol{\delta}_2)^T(\mathbf{Y}_2 - \mathbf{X}(\omega)\boldsymbol{\delta}_2) \tag{6}$$

where $\mathbf{Y}_k = (y_k(1), \ldots, y_k(n))^T$, $\boldsymbol{\delta}_k = (A_k, B_k)^T$ for $k = 1, 2$ and

$$\mathbf{X}^T(\omega) = \begin{pmatrix} \cos(\omega) & \cos(2\omega) & \cdots & \cos(n\omega) \\ \sin(\omega) & \sin(2\omega) & \cdots & \sin(n\omega) \end{pmatrix}.$$

Minimizing (6) with respect to $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ for a given ω, we obtain

$$\widetilde{\boldsymbol{\delta}}_k(\omega) = (\mathbf{X}^T(\omega)\mathbf{X}(\omega))^{-1}\mathbf{X}^T(\omega)\mathbf{Y}_k, \quad k = 1, 2. \tag{7}$$

Replacing $\boldsymbol{\delta}_k$ by $\widetilde{\boldsymbol{\delta}}_k(\omega)$ in $R(\boldsymbol{\xi}) = R(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \omega)$, we have

$$\begin{aligned} R(\widetilde{\boldsymbol{\delta}}_1(\omega), \widetilde{\boldsymbol{\delta}}_2(\omega), \omega) &= (\mathbf{Y}_1 - P_{\mathbf{X}(\omega)}\mathbf{Y}_1)^T(\mathbf{Y}_1 - P_{\mathbf{X}(\omega)}\mathbf{Y}_1) + (\mathbf{Y}_2 - P_{\mathbf{X}(\omega)}\mathbf{Y}_2)^T(\mathbf{Y}_2 - P_{\mathbf{X}(\omega)}\mathbf{Y}_2) \\ &= \mathbf{Y}_1^T(\mathbf{I} - P_{\mathbf{X}(\omega)})\mathbf{Y}_1 + \mathbf{Y}_2^T(\mathbf{I} - P_{\mathbf{X}(\omega)})\mathbf{Y}_2 = Q(\omega) \ \text{(say)}, \end{aligned}$$

where $P_{\mathbf{X}(\omega)} = \mathbf{X}(\omega)(\mathbf{X}^T(\omega)\mathbf{X}(\omega))^{-1}\mathbf{X}^T(\omega)$ is the projection matrix on the column space of $\mathbf{X}(\omega)$. Therefore, we can use a two step procedure; first estimate ω by minimizing $Q(\omega)$ with respect to ω and denote it as $\widetilde{\omega}$; then estimate A_k and B_k, $k = 1, 2$ using $\begin{pmatrix} \widetilde{A}_k \\ \widetilde{B}_k \end{pmatrix} = \widetilde{\boldsymbol{\delta}}_k(\widetilde{\omega})$. The procedure described above is basically the separable regression technique given by Richards (Richards[9]). If we have m channels with single frequency, say ω, then one can minimize a similar term as $Q(\omega)$ with sum of m terms. The details are given in Section 5.

Under Assumption 1, $e_1(t) \overset{i.i.d}{\sim} (0, \sigma_1^2)$, $e_2(t) \overset{i.i.d}{\sim} (0, \sigma_2^2)$ with correlation $\dfrac{\sigma_{12}}{\sigma_1\sigma_2}$. According to Assumption 2, A_i and B_i are not simultaneously equal to zero for $i = 1, 2$. This implies that the frequency ω is present in both the channels. We prove the strong consistency under these assumptions and is stated in the following theorem.

Once, we have estimated the frequency ω and linear parameters A_k and B_k, $k = 1, 2$ using the least squares method, the estimators of σ_1^2, σ_2^2 and σ_{12} are obtained as follows;

$$\widetilde{\sigma}_j^2 = \frac{1}{n}\sum_{t=1}^{n}\Big[y_j(t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big]^2, \quad j = 1, 2, \tag{8}$$

$$\widetilde{\sigma}_{12} = \frac{1}{n}\sum_{t=1}^{n}\Big[y_1(t) - \widetilde{A}_1\cos(\widetilde{\omega}t) - \widetilde{B}_1\sin(\widetilde{\omega}t)\Big]\Big[y_2(t) - \widetilde{A}_2\cos(\widetilde{\omega}t) - \widetilde{B}_2\sin(\widetilde{\omega}t)\Big]. \tag{9}$$

The LSEs of $\boldsymbol{\beta} = (\boldsymbol{\delta}_1^T, \boldsymbol{\delta}_2^T)$ and ω, obtained above, are strongly consistent and have asymptotically normally distributed. The results are stated in following theorems. Theorems 2.1, 2.2 and 2.3 are proved in Appendix A, Appendix B and Appendix C, respectively.

Theorem 2.1. *Under Assumptions 1 and 2, $\widetilde{A}_k$, $\widetilde{B}_k$, $k = 1, 2$ and $\widetilde{\omega}$ which minimizes*

$$\sum_{j=1}^{2}\sum_{t=1}^{n}[y_j(t) - A_j\cos(\omega t) - B_j\sin(\omega t)]^2$$

are strongly consistent estimators. ∎

Theorem 2.2. *Under the same assumptions as Theorem 2.1, as $n \to \infty$,*

$$\left(n^{\frac{1}{2}}(\widetilde{A}_1 - A_1^0), n^{\frac{1}{2}}(\widetilde{B}_1 - B_1^0), n^{\frac{1}{2}}(\widetilde{A}_2 - A_2^0), n^{\frac{1}{2}}(\widetilde{B}_2 - B_2^0), n^{\frac{3}{2}}(\widetilde{\omega} - \omega^0)\right) \xrightarrow{d} \mathcal{N}_5(\mathbf{0}, \mathbf{\Gamma}^{-1}\mathbf{G}\mathbf{\Gamma}^{-1})$$

where
$\mathbf{G} = \begin{pmatrix} \mathbf{\Sigma}\otimes\mathbf{I}_2 & \mathbf{v} \\ \mathbf{v}^T & \psi_2 \end{pmatrix}$ *with* $\psi_2 = \dfrac{2}{3}\Big[\sigma_1^2({A_1^0}^2 + {B_1^0}^2) + \sigma_2^2({A_2^0}^2 + {B_2^0}^2) + 2\sigma_{12}(A_1^0A_2^0 + B_1^0B_2^0)\Big]$,

$$\mathbf{v}^T = (B_1^0\sigma_1^2 + B_2^0\sigma_{12}, -A_1^0\sigma_1^2 - A_2^0\sigma_{12}, B_2^0\sigma_2^2 + B_1^0\sigma_{12}, -A_2^0\sigma_2^2 - A_1^0\sigma_{12})$$

and
$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{u} \\ \mathbf{u}^T & \phi_2 \end{pmatrix}$ *with* $\phi_2 = \frac{1}{3}({A_1^0}^2 + {B_1^0}^2 + {A_2^0}^2 + {B_2^0}^2)$, $\mathbf{u}^T = \left(\dfrac{B_1^0}{2}, -\dfrac{A_1^0}{2}, \dfrac{B_2^0}{2}, -\dfrac{A_2^0}{2}\right)$.

∎

The matrix $\mathbf{\Gamma}^{-1}$ has the following form. Write $\rho_s = \sum_{j=1}^{2}({A_j^0}^2 + {B_j^0}^2)$, then

$$\mathbf{\Gamma}^{-1} = \begin{pmatrix} 1+\frac{3{B_1^0}^2}{\rho_s} & -\frac{3A_1^0B_1^0}{\rho_s} & \frac{3B_1^0B_2^0}{\rho_s} & -\frac{3A_2^0B_1^0}{\rho_s} & -\frac{6B_1^0}{\rho_s} \\ -\frac{3A_1^0B_1^0}{\rho_s} & 1+\frac{3{A_1^0}^2}{\rho_s} & -\frac{3A_1^0B_2^0}{\rho_s} & \frac{3A_1^0A_2^0}{\rho_s} & \frac{6A_1^0}{\rho_s} \\ \frac{3B_1^0B_2^0}{\rho_s} & -\frac{3A_1^0B_2^0}{\rho_s} & 1+\frac{3{B_2^0}^2}{\rho_s} & -\frac{3A_2^0B_2^0}{\rho_s} & -\frac{6B_2^0}{\rho_s} \\ -\frac{3A_2^0B_1^0}{\rho_s} & \frac{3A_1^0A_2^0}{\rho_s} & -\frac{3A_2^0B_2^0}{\rho_s} & 1+\frac{3{A_2^0}^2}{\rho_s} & \frac{6A_2^0}{\rho_s} \\ -\frac{6B_1^0}{\rho_s} & \frac{6A_1^0}{\rho_s} & -\frac{6B_2^0}{\rho_s} & \frac{6A_2^0}{\rho_s} & \frac{12}{\rho_s} \end{pmatrix}.$$

Theorem 2.3. *Under the same assumptions as Theorem 2.1, $\widetilde{\sigma}_1^2$, $\widetilde{\sigma}_2^2$ and $\widetilde{\sigma}_{12}$ are strongly consistent estimators of σ_1^2, σ_2^2 and σ_{12}, respectively.*

Remark 1. The asymptotic variances of the LSEs of the linear parameters as well as ω depends on the true values of the linear parameters from all the channels through ρ_s whereas the asymptotic variance-covariance matrix $\mathbf{\Gamma}^{-1}\mathbf{G}\mathbf{\Gamma}^{-1}$ does not depend on the true value of ω.

3. Generalized Least Squares Estimator

In the previous section we have discussed about the LSEs of the unknown parameters of the two-channel sinusoidal model (1). It is observed that although the LSEs are consistent, they are not efficient in the sense when errors are from bivariate normal distributions, the asymptotic variances of the LSEs do not attain the Cramer-Rao lower bound. In this section, we discuss the generalized LSE of the unknown parameters for a two-channel sinusoidal model. It is observed that the generalized LSEs are consistent and when the errors are from bivariate normal distribution and if the noise covariance matrix $\mathbf{\Sigma}$ is known, the asymptotic variances of these estimators attain the Cramer-Rao lower bound.

First it is assumed that the noise covariance matrix $\mathbf{\Sigma}$ is known. The generalized LSEs of the unknown parameters can be obtained by minimizing $S(\boldsymbol{\beta}, \omega)$, where

$$\frac{1}{|\mathbf{\Sigma}|} S(\boldsymbol{\beta}, \omega) = \sum_{t=1}^{n} (\mathbf{y}(t) - \boldsymbol{\mu}(t))^T \mathbf{\Sigma}^{-1} (\mathbf{y}(t) - \boldsymbol{\mu}(t)) \tag{10}$$

with respect to the elements of $\boldsymbol{\beta} = (A_1, B_1, A_2, B_2)$ and ω. Here $\mathbf{y}(t)$ is the observation vector at time point t and $E[\mathbf{y}(t)] = \boldsymbol{\mu}(t, \boldsymbol{\beta}, \omega) \equiv \boldsymbol{\mu}(t)$. The $\mathbf{\Sigma}$ matrix is same as defined in Assumption 1. Note that when the errors are from bivariate normal distribution with mean vector zero, and dispersion matrix $\mathbf{\Sigma}$, then the GLSEs become the maximum likelihood estimators also. This is the main motivation of considering the GLSEs.

Observe that $S(\boldsymbol{\beta}, \omega)$ can be written as

$$\begin{aligned} S(\boldsymbol{\beta}, \omega) &= \sum_{t=1}^{n} \Big[\sigma_2^2 \Big(y_1(t) - \mu_1(t) \Big)^2 + \sigma_1^2 \Big(y_2(t) - \mu_2(t) \Big)^2 \\ &\qquad -2\sigma_{12} \Big(y_1(t) - \mu_1(t) \Big) \Big(y_2(t) - \mu_2(t) \Big) \Big]. \end{aligned}$$

Here

$$\mu_k(t) = A_k \cos(\omega t) + B_k \sin(\omega t), \qquad \text{for} \qquad k = 1, 2,$$

are two elements of the mean vector $\boldsymbol{\mu}(t)$.

Suppose $\widehat{\boldsymbol{\xi}} = (\widehat{A}_1, \widehat{B}_1, \widehat{A}_2, \widehat{B}_2, \widehat{\omega})$ minimizes $S(\boldsymbol{\beta}, \omega) = S(\boldsymbol{\xi})$, then $\widehat{\boldsymbol{\xi}}$ is called the generalized LSEs of $\boldsymbol{\xi}^0$. The following theorems provide the consistency and asymptotic normality properties of $\widehat{\boldsymbol{\xi}}$. Theorem 3.1 is proved in Appendix D and Theorem 3.2 is in Appendix E.

Theorem 3.1. *Suppose the vector of sequence of error $\{\mathbf{e}(t)\}$ from model (1) satisfies Assumption 1 and the elements of $\mathbf{\Sigma}$ matrix, σ_1^2, σ_2^2 and σ_{12} are known. Then, $\widehat{\boldsymbol{\xi}}$ is a strongly consistent estimator of $\boldsymbol{\xi}^0$.*

Theorem 3.2. *Under the same assumptions as Theorem 3.1, as $n \to \infty$,*

$$(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}^0)\mathbf{D_1}^{-1} \xrightarrow{d} \mathcal{N}_5(\mathbf{0}, 2\boldsymbol{\Gamma}_g^{-1}), \tag{11}$$

where

$$\boldsymbol{\Gamma}_g = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{a} \\ \boldsymbol{a}^T & b \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{\Sigma}_{11} &= \left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right), \quad \mathbf{D}_1 = diag\left\{n^{\frac{1}{2}}, n^{\frac{1}{2}}, n^{\frac{1}{2}}, n^{\frac{1}{2}}, n^{\frac{3}{2}}\right\}, \\ \boldsymbol{a}^T &= \frac{1}{2|\boldsymbol{\Sigma}|}\begin{pmatrix} B_1\sigma_2^2 - B_2\sigma_{12} & A_2\sigma_{12} - A_1\sigma_2^2 & B_2\sigma_1^2 - B_1\sigma_{12} & A_1\sigma_{12} - A_2\sigma_1^2 \end{pmatrix}, \\ b &= \frac{1}{3|\boldsymbol{\Sigma}|}\left[\sigma_2^2(A_1^2 + B_1^2) + \sigma_1^2(A_2^2 + B_2^2) - 2\sigma_{12}(A_1A_2 + B_1B_2)\right]. \end{aligned}$$

Here '$\otimes$' denotes the Kronecker product.

Remark 2. Similar to the LSEs, the asymptotic distribution of $\widehat{\boldsymbol{\xi}}$ depends of the true values of the linear parameters and $\boldsymbol{\Sigma}$ matrix and does not depend of the frequency.

By inverting $\boldsymbol{\Gamma}_g$ matrix, the asymptotic variances of the generalized LSEs can be obtained. The asymptotic variances are

$$\mathrm{Var}\left(\widehat{A}_i\right) = \frac{2}{n}\left(\sigma_i^2 + \frac{B_i^2}{b}\right), \quad \mathrm{Var}\left(\widehat{B}_i\right) = \frac{2}{n}\left(\sigma_i^2 + \frac{A_i^2}{b}\right), \quad i = 1, 2 \text{ and } \mathrm{Var}\left(\widehat{\omega}\right) = \frac{8}{n^3 b},$$

where b is same as defined in Theorem 3.2.

Now first we provide the method of implementation of the generalized LSEs when the matrix $\boldsymbol{\Sigma}$ is known, and then we discuss the case when $\boldsymbol{\Sigma}$ is unknown. First observe that for fixed ω, the generalized LSE of $\boldsymbol{\beta}$, say $\widehat{\boldsymbol{\beta}}(\omega)$, can be obtained by solving

$$\frac{\partial}{\partial A_1}S(\boldsymbol{\beta}, \omega) = 0, \quad \frac{\partial}{\partial B_1}S(\boldsymbol{\beta}, \omega) = 0, \quad \frac{\partial}{\partial A_2}S(\boldsymbol{\beta}, \omega) = 0, \quad \frac{\partial}{\partial B_2}S(\boldsymbol{\beta}, \omega) = 0. \tag{12}$$

After some manipulations, the four equations in (12) can be written in a matrix form as given below:

$$\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{M}_n(\omega)\right)\boldsymbol{\beta}^T = \left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right)\boldsymbol{W}_n(\omega), \tag{13}$$

here

$$\boldsymbol{M}_n(\omega) = \frac{2}{n}\begin{bmatrix} \sum_{t=1}^n \cos^2(\omega t) & \sum_{t=1}^n \cos(\omega t)\sin(\omega t) \\ \sum_{t=1}^n \cos(\omega t)\sin(\omega t) & \sum_{t=1}^n \sin^2(\omega t) \end{bmatrix},$$

$$\boldsymbol{W}_n(\omega) = \frac{2}{n}\begin{bmatrix} \sum_{t=1}^{n} y_1(t)\cos(\omega t) \\ \sum_{t=1}^{n} y_1(t)\sin(\omega t) \\ \sum_{t=1}^{n} y_2(t)\cos(\omega t) \\ \sum_{t=1}^{n} y_2(t)\sin(\omega t) \end{bmatrix},$$

and $\boldsymbol{I}_2$ is the 2×2 identity matrix. Hence,

$$\begin{aligned} \widehat{\boldsymbol{\beta}}^T(\omega) &= \left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{M}_n(\omega)\right)^{-1}\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right)\boldsymbol{W}_n(\omega) \\ &= \left(\boldsymbol{\Sigma} \otimes (\boldsymbol{M}_n(\omega))^{-1}\right)\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right)\boldsymbol{W}_n(\omega) \\ &= \left(\mathbf{I}_2 \otimes (\boldsymbol{M}_n(\omega))^{-1}\right)\boldsymbol{W}_n(\omega). \end{aligned}$$

The last expression is independent of the $\boldsymbol{\Sigma}$ matrix and is exactly equal to the LSEs which for a given ω, minimizes the sums of the residual sum of squares from two channels. Therefore, the generalized LSE of ω^0 can be obtained by minimizing $S(\widehat{\boldsymbol{\beta}}(\omega), \omega)$ with respect to ω. If $\widehat{\omega}$ is the generalized LSE of ω^0, then $\widehat{\boldsymbol{\beta}}$, the generalized LSE of $\boldsymbol{\beta}^0$ can be obtained as $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}(\widehat{\omega})$. Although, the two-channel sinusoidal model has five signal parameters, the generalized LSEs of the unknown parameters can be obtained by solving only one one-dimensional optimization problem.

Remark 3. Observe that for large n, $\boldsymbol{M}_n(\omega) = \boldsymbol{I}_2 + o(1/n)$. Hence for large n,

$$\begin{aligned} \widehat{\boldsymbol{\beta}}^T(\omega) &= (\boldsymbol{\Sigma} \otimes \boldsymbol{I}_2)\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right)\boldsymbol{W}_n(\omega) + o_p(1/n) \\ &= \boldsymbol{W}_n(\omega) + o_p(1/n), \end{aligned}$$

the usual approximate LSE.

Remark 4. It has been assumed that in generalized least squares (GLS) method that the variance covariance matrix of $\mathbf{y}(t)$ is known, but in real life situation, it is generally unknown. We can use consistent estimators of σ_1^2, σ_2^2 and σ_{12} discussed in Section 2.

Remark 5. A special case will be if we ignore that σ_{12} is non-zero. Then, the GLS method boils down to the weighted least squares method in this case and minimizes

$$Q_W(\boldsymbol{\xi}) = \sum_{t=1}^{n}\frac{1}{\sigma_1^2}(y_1(t) - A_1\cos(\omega t) - B_1\sin(\omega t))^2 + \sum_{t=1}^{n}\frac{1}{\sigma_2^2}(y_1(t) - A_2\cos(\omega t) - B_2\sin(\omega t))^2, \tag{14}$$

with respect to $\boldsymbol{\xi}$.

4. Numerical Experiments

In the last two sections we demonstrated the LSEs and GLSEs and developed their asymptotic properties. In this section we would like to study the behavior of these estimators when the sample size is finite, based on simulation experiments. We report the results of these numerical experiments conducted for different model parameters and error variances. We consider model (1) with the following parameter values

$$A_1^0 = 1.5, \quad B_1^0 = 2.0, \quad A_2^0 = 3.0, \quad B_2^0 = 2.5, \quad \omega = 1.25. \tag{15}$$

The sequence of random vector $\{\mathbf{e}(t)\}$ is generated in different ways; 1) $\{\mathbf{e}(t)\}$ is a sequence of bivariate normal vectors with mean $\mathbf{0}$ and variance covariance matrix $\boldsymbol{\Sigma}$; 2) $\{\mathbf{e}(t)\}$ is a sequence of bivariate t-distribution with degrees of freedom 5 and variance covariance matrix $\boldsymbol{\Sigma}$; 3) $\{\mathbf{e}(t)\}$ is a sequence of bivariate t-distribution with degrees of freedom 10 with the same $\boldsymbol{\Sigma}$. The variance-covariance matrix considered for simulation studies are

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 3.0 & 0.95 \\ 0.95 & 3.0 \end{pmatrix}, \qquad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 2.0 & 0.9 \\ 0.9 & 2.0 \end{pmatrix}.$$

sample sizes are 100, 200, 300, 400 and 500. The number of replications used in these experiments is 5000. The data are generated from all the three cases and we calculate the average estimates and mean squared errors (MSES) using both the proposed estimation methods. We report the average estimates, MSEs and the corresponding asymptotic variances of both the estimators in Figures 1-10. In Figure 1, the average estimates of LSE and GLSE of A_1 from all the three cases considered here are plotted against the sample size along with a horizontal line at the true value of the parameter. The plot at the left is for dispersion matrix $\boldsymbol{\Sigma}_1$ and the right one is for $\boldsymbol{\Sigma}_2$. Similarly, the average estimates of B_1, A_2, B_2 and ω are plotted in Figures 2-5, respectively. The MSEs along with the corresponding asymptotic variances of LSE as well as the GLSE of A_1, B_1, A_2, B_2 and ω are plotted in Figures 6-10 respectively.

The following observation can be made from the numerical experiment considered here.

(1) From Figures 1-5 it is quite clear that the average estimates are close to the true values at least when sample sizes are grater than 100. The small sample biases in case of LSEs are slightly larger than the biases in case of GLSEs, specially for the linear parameter estimators. The biases of both LSEs and GLSEs for the frequency parameter are quite small for all the error distributions considered here. The biases of every parameter estimators decrease as the sample size increases.
(2) Both the estimation methods tend to underestimate the linear parameters for moderate sample sizes.

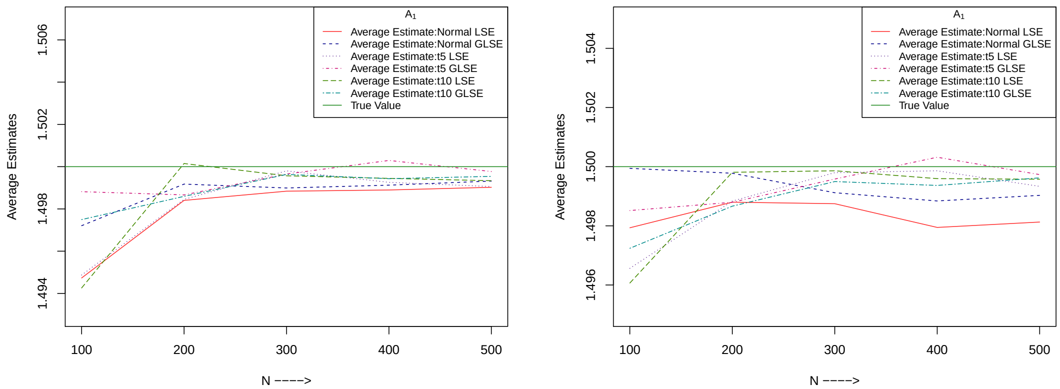

FIGURE 1. Average estimates of A_1 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

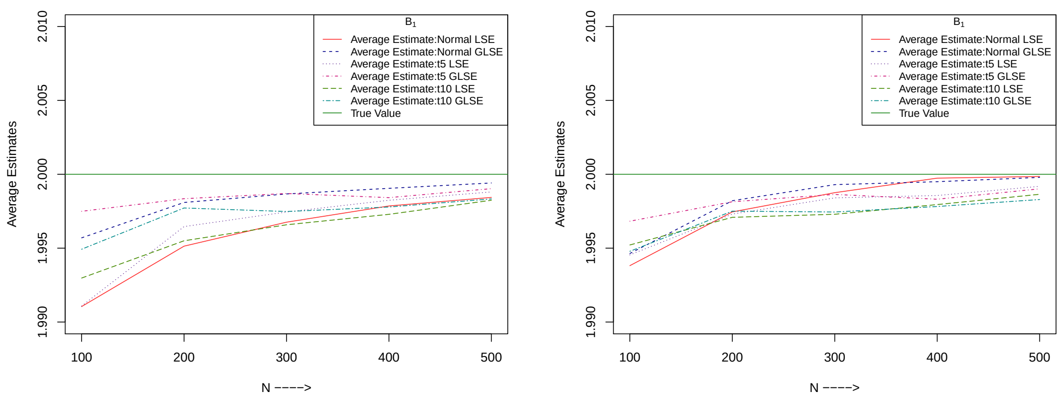

FIGURE 2. Average estimates of B_1 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

(3) In Figures 1-5, the spread of the average estimates in left plots are wider than that of the right plots. This is due to the fact that the elements of $\mathbf{\Sigma}_1$ are greater than the elements of $\mathbf{\Sigma}_2$.

(4) From Figures 6-10 it is quite clear that the MSEs of the LSEs as well as the GLSEs decrease as the sample size increases.

(5) The MSEs of the GLSEs are smaller than the MSEs of the LSEs and the difference between them decreases as the sample size increases.

(6) The MSEs are quite close to the corresponding asymptotic variances in most of the cases considered here.

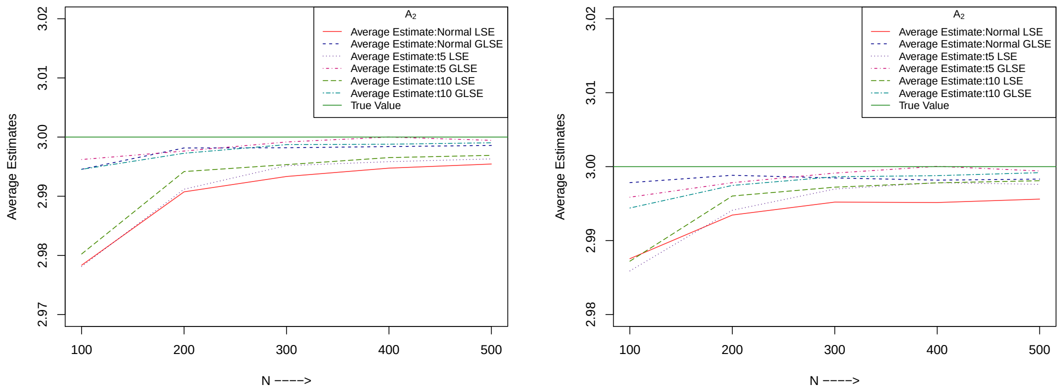

FIGURE 3. Average estimates of A_2 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

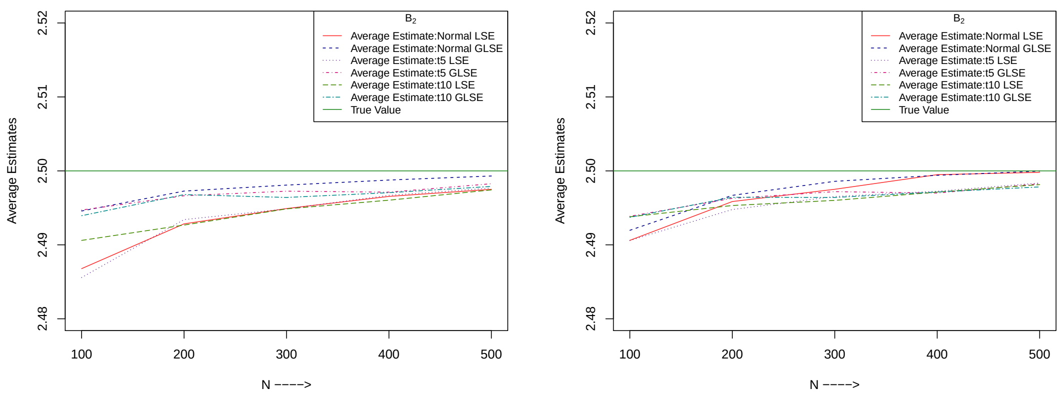

FIGURE 4. Average estimates of B_2 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

(7) In MSE and asymptotic variance plots, there are clear separation of lines for LSE and GLSE. The MSEs in all three cases (i), (ii) and (iii) stated above for any particular estimators using LS method are clubbing at slightly larger values than the case when GLS method are used. The lines for GLS method are also clubbed.

(8) From Figures 1-10, it is observed that although the effect of error distribution is quite prominent in case of biases, it is not felt that much in case of MSEs of the estimates.

In a separate set-up, we have conducted experiments to observe the nature of change of MSEs and asymptotic variances as the correlation coefficient ρ between $\{y_1(t)\}$ and $\{y_2(t)\}$ changes. It can be shown when $A_1 = A_2$, $B_1 = B_2$ and $\sigma_1^2 = \sigma_2^2$, the asymptotic variances

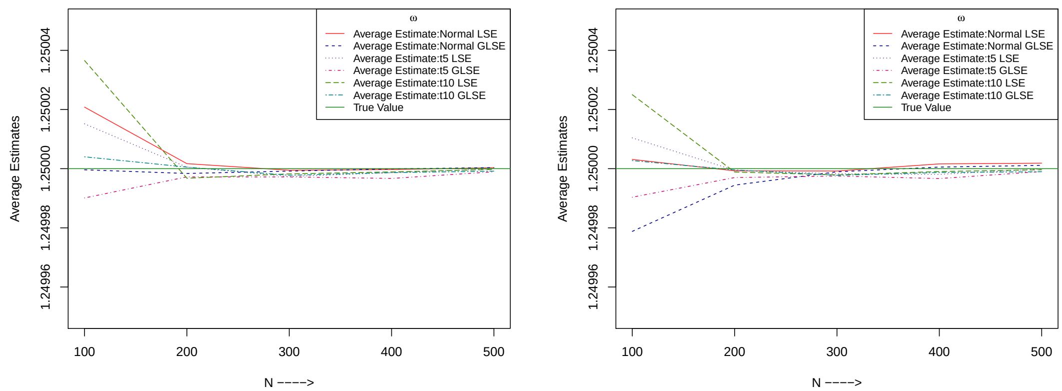

FIGURE 5. Average estimates of ω of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

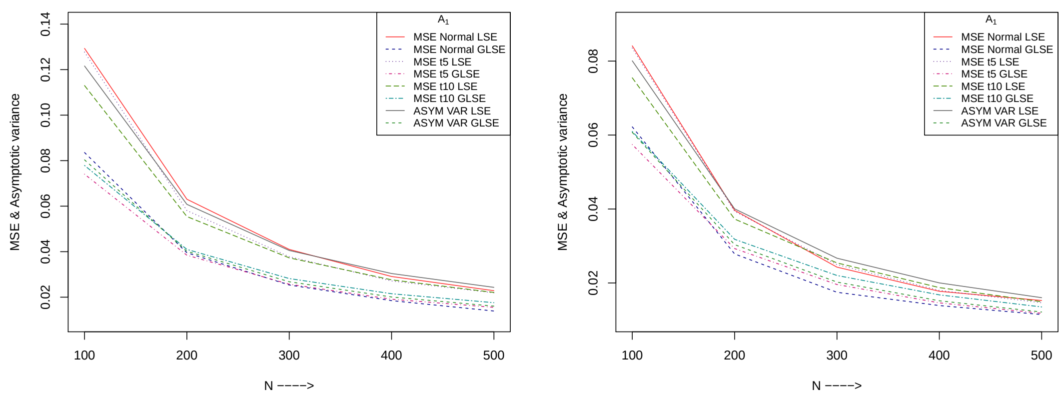

FIGURE 6. MSE and asymptotic variance of estimators of A_1 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

of the GLSEs are linear functions of ρ which is not the case in general. In this special case, i.e. when $A_1 = A_2$, $B_1 = B_2$ and $\sigma_1^2 = \sigma_2^2$, it implies that same signals are coming from two different channels and they are correlated. Consider the following two cases:

I: Model (15) with $\mathbf{\Sigma} = \begin{pmatrix} 2.0 & 2\rho \\ 2\rho & 2.0 \end{pmatrix}$, where ρ is the correlation coefficient between $\{y_1(t)\}$ and $\{y_2(t)\}$.

II: $A_1 = A_2 = 2.0$, $B_1 = B_2 = 4.0$ and same $\mathbf{\Sigma}$ as in I, so that $\sigma_1^2 = \sigma_2^2 = 2.0$.

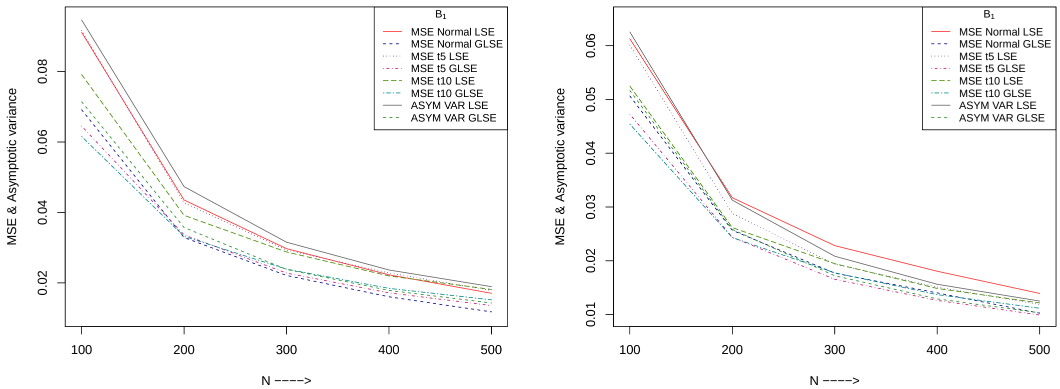

FIGURE 7. MSE and asymptotic variance of estimators of B_1 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

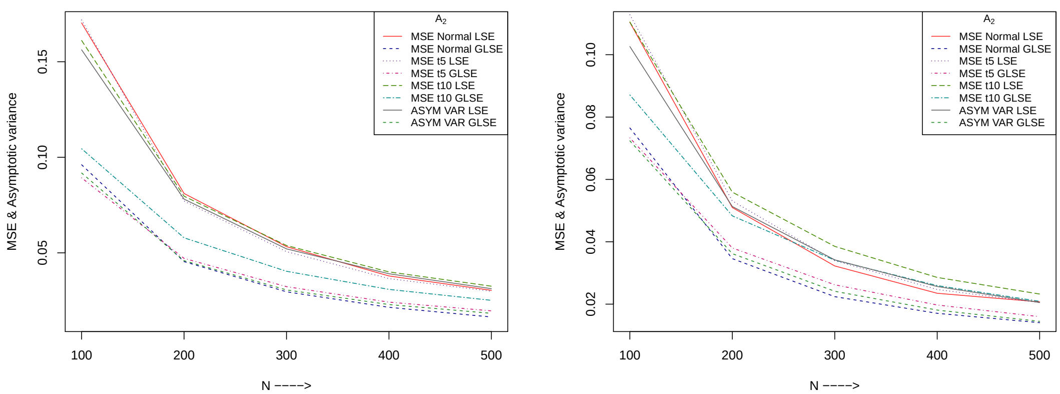

FIGURE 8. MSE and asymptotic variance of estimators of A_2 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

The sample size is fixed at $N = 500$ and ρ is varied from 0 to 0.95. The MSEs and asymptotic variances of the LSE and the GLSE of ω have been plotted in the same figure. For I and II, these are plotted in Figure 11 and Figure 12, respectively. We observe in Figure 11 that the asymptotic variance as well as the MSE of the LSE of ω increase as the correlation coefficient ρ increases. Whereas in case of GLSE, both the MSE and the asymptotic variance first increase and then decrease as ρ increases. This pattern has been observed in case of other parameter estimators. In case of Figure 12, for the LSE as well as the GLSE, both the MSEs and the asymptotic variances increase as ρ increases. We also notice that in this particular case, the asymptotic variances of LSE and GLSE coincide whereas the MSE of GLSE is smaller than the MSE of the LSE.

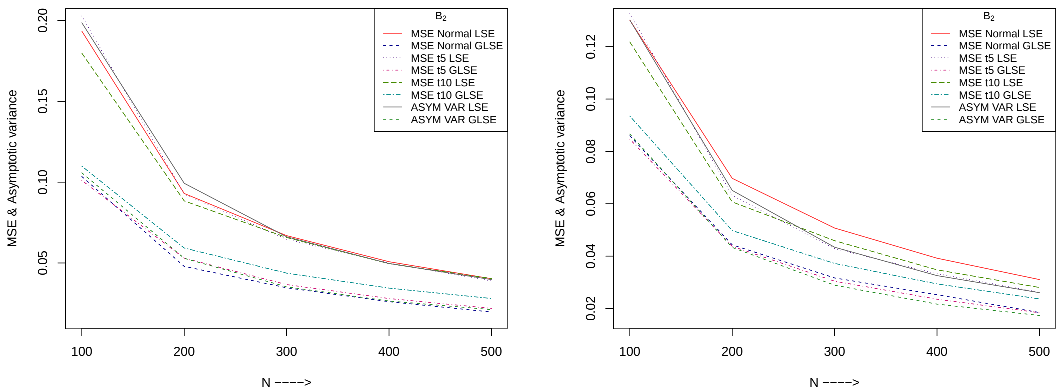

FIGURE 9. MSE and asymptotic variance of estimators of B_2 of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

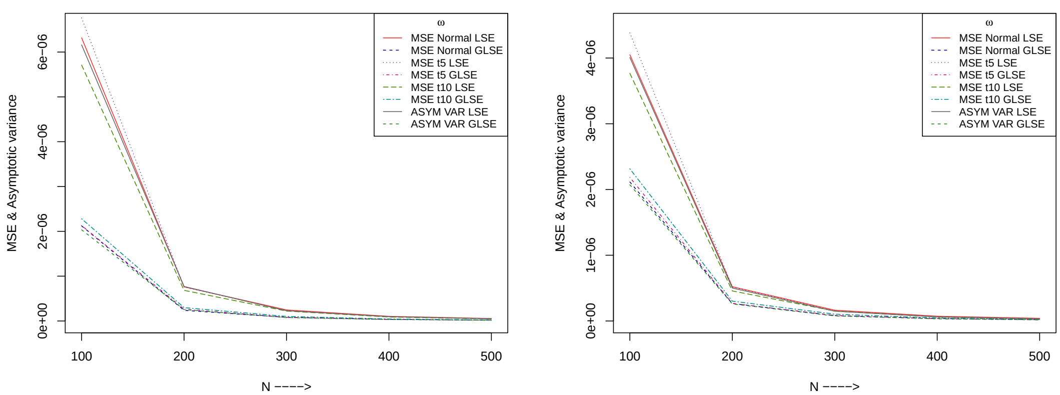

FIGURE 10. MSE and asymptotic variance of estimators of ω of model (15) when variance-covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

5. Multichannel Sinusoidal Model with m Channels

In previous sections, we have discussed the LSEs and proposed the GLSEs to estimate the unknown parameters for a model with two channels. In this section, we consider a single frequency model with m channels. We first define the model in the same line as model (1)

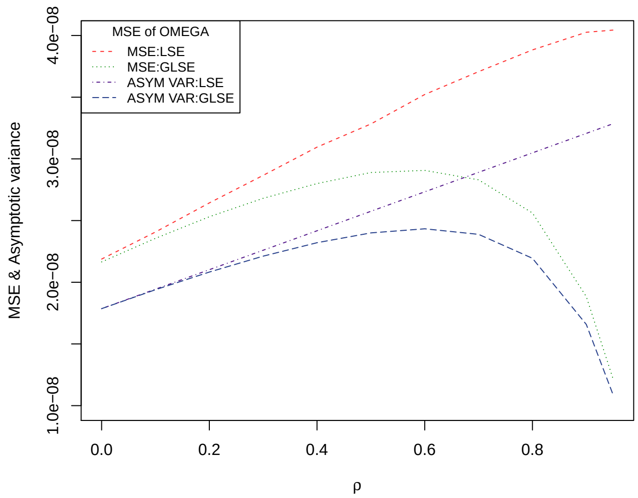

FIGURE 11. MSE and asymptotic variance of LSE and GLSE of ω as ρ increases for model I.

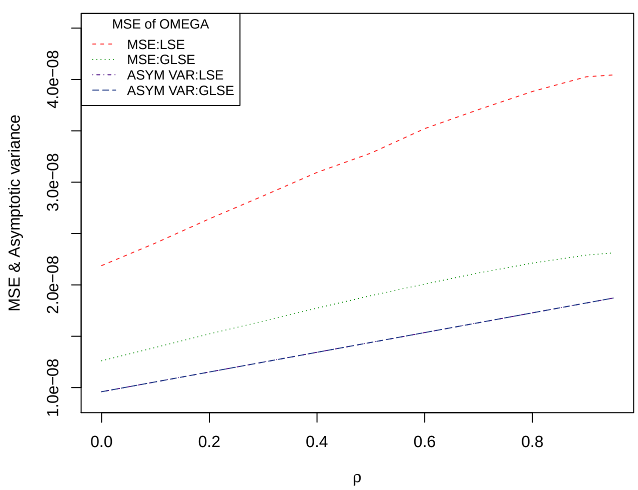

FIGURE 12. MSE and asymptotic variance of LSE and GLSE of ω as ρ increases for model II.

and state the required assumptions. The signal frequency m-channel model is defined as

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{pmatrix} = \begin{pmatrix} A_1^0 & B_1^0 \\ A_2^0 & B_2^0 \\ \vdots & \vdots \\ A_m^0 & B_m^0 \end{pmatrix} \begin{pmatrix} \cos(\omega^0 t) \\ \sin(\omega^0 t) \end{pmatrix} + \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_m(t) \end{pmatrix}. \qquad (16)$$

In matrix notation, model (16) can be written similarly as model (1)

$$\mathbf{y}(t) = \mathbf{A}^0\boldsymbol{\theta}(\omega^0, t) + \mathbf{e}(t) \tag{17}$$

$$= \boldsymbol{\mu}(t; \boldsymbol{\beta}^0, \omega^0) + \mathbf{e}(t). \tag{18}$$

Here $\mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T$, $\mathbf{e}(t) = (e_1(t), e_2(t), \ldots, e_m(t)))^T$, $\boldsymbol{\theta}(\omega^0, t)$ is same as defined in Section 1, $\boldsymbol{\beta}^0 = (A_1^0, B_1^0, A_2^0, B_2^0, \ldots, A_m^0, B_m^0)$, the vector of linear parameters of order $2m$ and $\mathbf{A}^0$ is an $m \times 2$ matrix of the following form;

$$\mathbf{A}^0 = \begin{pmatrix} A_1^0 & B_1^0 \\ A_2^0 & B_2^0 \\ \vdots & \vdots \\ A_m^0 & B_m^0 \end{pmatrix}.$$

The mean component of the vector valued signal $\mathbf{y}(t)$ is $\boldsymbol{\mu}(t; \boldsymbol{\beta}^0, \omega^0) = (\mu_1(t; \boldsymbol{\beta}^0, \omega^0), \mu_2(t; \boldsymbol{\beta}^0, \omega^0),$ $\ldots, \mu_m(t; \boldsymbol{\beta}^0, \omega^0))^T$. The signal from the k-th channel, takes the following form;

$$y_k(t) = \Big[A_k^0 \cos(\omega^0 t) + B_k^0 \sin(\omega^0 t)\Big] + e_k(t), \quad t = 1, \ldots, n$$

where $k = 1, 2, \ldots, m$. Model (16) being a single component m-channel model, $\omega^0 \in (0, \pi)$ is the common frequency, A_k^0 and B_k^0 are the amplitudes attached to the k-th channel. The error vector $\mathbf{e}(t)$ represents the noise part and its k-th component corresponds to the additive noise of the kth channel signal and components of $\mathbf{e}(t)$ are correlated. The following assumptions are required for development of properties of the estimators.

Assumption 3. *The sequence of m-variate random vector $\{\mathbf{e}(t)\}$ are i.i.d. with mean vector* $\mathbf{0}$ *and covariance matrix*

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{1m} & \sigma_{2m} & \sigma_{3m} & \cdots & \sigma_m^2 \end{pmatrix}.$$

Assumption 4. *For $k = 1, 2, \ldots, m$, A_k^0 and B_k^0 are not simultaneously equal to zero.*

Write $\boldsymbol{\eta} = (A_1, B_1, A_2, B_2, \ldots, A_m, B_m, \omega)^T$ and suppose $\boldsymbol{\eta}^0$ denote the true value of $\boldsymbol{\eta}$. Then the LSE, $\widetilde{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ minimizes

$$\begin{aligned} R(\boldsymbol{\eta}) &= \sum_{t=1}^{n} \mathbf{e}^T(t)\mathbf{e}(t) = \sum_{t=1}^{n} [e_1^2(t) + e_2^2(t) + \cdots + e_m^2(t)] \\ &= \sum_{t=1}^{n}\sum_{k=1}^{m} [y_k(t) - A_k\cos(\omega t) - B_k \sin(\omega t)]^2 \\ &= \sum_{k=1}^{m} (\mathbf{Y}_k - \mathbf{X}(\omega)\boldsymbol{\delta}_k)^T(\mathbf{Y}_k - \mathbf{X}(\omega)\boldsymbol{\delta}_k), \end{aligned}$$

where $\mathbf{Y}_k$, $\boldsymbol{\delta}_k$ and $\mathbf{X}(\omega)$ are same as defined in Section 2.

Now using separable regression technique, we can write for a given ω

$$R(\boldsymbol{\delta}_1(\omega), \ldots, \boldsymbol{\delta}_m(\omega), \omega) = \sum_{k=1}^{m} \mathbf{Y}_k^T(\mathbf{I} - \mathbf{P}_{\mathbf{X}(\omega)})\mathbf{Y}_k. \tag{19}$$

Therefore, ω is estimated by minimizing (19) and then the linear parameter vector $\boldsymbol{\delta}_k$ for the k-th channel using (7). We denote these estimators as $\widetilde{A}_1, \widetilde{B}_1, \ldots, \widetilde{A}_m, \widetilde{B}_m$ and $\widetilde{\omega}$ which are the elements $\widetilde{\boldsymbol{\eta}}$. The strong consistency of $\widetilde{\boldsymbol{\eta}}$ can be proved similarly as the proof of Theorem 2.1 and are not discussed here. We state the asymptotic distribution here without the proof.

Theorem 5.1. *Under Assumptions 3 and 4, as $n \to \infty$*

$$\big(n^{\frac{1}{2}}(\widetilde{A}_1 - A_1^0), n^{\frac{1}{2}}(\widetilde{B}_1 - B_1^0), \ldots, n^{\frac{1}{2}}(\widetilde{A}_m - A_m^0), n^{\frac{1}{2}}(\widetilde{B}_m - B_m^0), n^{\frac{3}{2}}(\widetilde{\omega} - \omega^0)\big) \xrightarrow{d} \mathcal{N}_{2m+1}(\mathbf{0}, \boldsymbol{\Gamma}_m^{-1}\mathbf{G}_m\boldsymbol{\Gamma}_m^{-1}).$$

Here $\boldsymbol{\Gamma}_m = \begin{pmatrix} \mathbf{I}_{2m} & \boldsymbol{u}_m \\ \boldsymbol{u}_m^T & \phi_m \end{pmatrix}$, $\mathbf{I}_{2m}$ *being the identity matrix of order* $2m$, $\phi_m = \dfrac{1}{3}\displaystyle\sum_{k=1}^{m}({A_k^0}^2 + {B_k^0}^2)$ *and* $\boldsymbol{u}_m^T = \Big(\dfrac{B_1}{2}, -\dfrac{A_1}{2}, \cdots, \dfrac{B_m}{2}, -\dfrac{A_m}{2}\Big)$. *The matrix* $\mathbf{G}_m$ *has the following form;*

$$\mathbf{G}_m = \begin{pmatrix} \boldsymbol{\Sigma} \otimes \mathbf{I}_2 & \boldsymbol{v}_m \\ \boldsymbol{v}_m^T & \zeta \end{pmatrix}$$

where $\zeta = \dfrac{2}{3}\Big[\displaystyle\sum_{k=1}^{m}({A_k^0}^2 + {B_k^0}^2)\sigma_k^2 + \sum_{j=1}^{m}\sum_{k=1}^{m}(A_j^0A_k^0 + B_j^0B_k^0)\sigma_{jk}\Big]$ *and*

$$\boldsymbol{v}_m = (v_i), \quad \textit{with } v_{2i-1} = B_i\sigma_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{m} B_j\sigma_{ij} \;\; \textit{and } v_{2i} = -A_i\sigma_i^2 - \sum_{\substack{j=1\\ j\neq i}}^{m} A_j\sigma_{ij}, \quad i = 1, \ldots, m.$$

Once the linear parameters and the frequency are estimated using the LS method, the error variances and covariances (the elements of the $\mathbf{\Sigma}$ matrix) can be similarly obtained as the two -hannel model. Explicitly, they can be estimated as

$$\begin{aligned}\widetilde{\sigma}_j^2 &= \frac{1}{n}\sum_{t=1}^{n}\Big[y_j(t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big]^2, \quad j = 1,\ldots,m,\\ \widetilde{\sigma}_{jk} &= \frac{1}{n}\sum_{t=1}^{n}\Big[y_j(t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big]\Big[y_k(t) - \widetilde{A}_k\cos(\widetilde{\omega}t) - \widetilde{B}_k\sin(\widetilde{\omega}t)\Big].\end{aligned}$$

These estimators can be shown to be consistent similarly as Theorem 2.3.

In the following, we now discuss the GLSEs of the parameters of an m-channel single frequency sinusoidal model. Similar to the two-channel model, we assume that the variance-covariance matrix $\mathbf{\Sigma}$ is known. The GLSEs of the unknown parameters can be obtained by minimizing $S_m(\boldsymbol{\beta},\omega)$ where

$$S_m(\boldsymbol{\beta},\omega) = |\mathbf{\Sigma}|\sum_{t=1}^{n}\Big(\mathbf{y}(t) - \boldsymbol{\mu}(t;\boldsymbol{\beta},\omega)\Big)^T\mathbf{\Sigma}^{-1}\Big(\mathbf{y}(t) - \boldsymbol{\mu}(t;\boldsymbol{\beta},\omega)\Big)$$

with respect to the elements of $\boldsymbol{\beta} = (A_1, B_1, \ldots, A_m, B_m)$ and ω. Here, $\boldsymbol{\mu}(t;\boldsymbol{\beta},\omega)$ is an m-vector with k-th element

$$\mu_k(t; A_k, B_k, \omega) = A_k\cos(\omega t) + B_k\sin(\omega t).$$

Suppose $\widehat{\boldsymbol{\eta}} = (\widehat{A}_1, \widehat{B}_1, \ldots, \widehat{A}_m, \widehat{B}_m, \widehat{\omega})^T$ is the argument minimizer of $S_m(\boldsymbol{\eta}) = S_m(\boldsymbol{\beta},\omega)$. The strong consistency of $\widehat{\boldsymbol{\eta}}$ can be proved along the same line as Theorem 3.1 when the error vector is i.i.d. with mean vector zero and known $p\times p$ variance-covariance matrix $\mathbf{\Sigma}$. Under the same assumption $\widehat{\boldsymbol{\eta}}$ has the following asymptotic distribution.

Theorem 5.2. *Suppose that the sequence of error vectors $\{\mathbf{e}(t)\}$ is i.i.d. with mean zero and dispersion matrix $\mathbf{\Sigma}$, which is known. Then*

$$\big(n^{\frac{1}{2}}(\widehat{A}_1 - A_1^0), n^{\frac{1}{2}}(\widehat{B}_1 - B_1^0), \ldots, n^{\frac{1}{2}}(\widehat{A}_m - A_m^0), n^{\frac{1}{2}}(\widehat{B}_m - B_m^0), n^{\frac{3}{2}}(\widehat{\omega} - \omega^0)\big) \xrightarrow{d} \mathcal{N}_{2m+1}(\mathbf{0}, 2\mathbf{\Pi}_g^{-1})$$

where

$$\mathbf{\Pi}_g = \begin{pmatrix}\mathbf{\Sigma}^{-1}\otimes\mathbf{I}_2 & \boldsymbol{a}_m\\ \boldsymbol{a}_m^T & b_m\end{pmatrix}$$

with $\boldsymbol{a}_m = \Big(\sum_{j=1}^{m} B_j\sigma^{1j}, -\sum_{j=1}^{m} A_j\sigma^{1j}, \sum_{j=1}^{m} B_j\sigma^{2j}, -\sum_{j=1}^{m} A_j\sigma^{2j}, \ldots, \sum_{j=1}^{m} B_j\sigma^{mj}, -\sum_{j=1}^{m} A_j\sigma^{mj}\Big)$, *a vector of order* $2m$ *and* $b_m = \frac{1}{3}\sum_{j=1}^{m}\sigma^{jj}(A_j^{0\,2} + B_j^{0\,2}) + \frac{2}{3}\sum_{j<}^{m}\sum_{k}^{m}\sigma^{jk}(A_j^0A_k^0 + B_j^0B_k^0)$*. Here,* σ^{ij} *is the* (i,j)*-th element of* $\boldsymbol{\Sigma}^{-1}$ *matrix.*

We would like to emphasize here that the criterion function in LS optimization involves a single nonlinear parameter, that is , the frequency parameter ω. Thus it requires only an one dimensional optimization. Rest of the parameters are linear in nature and are directly estimated channel-wise. Therefore, the computational complexity is not significantly increased for more number of channels. In GLS method, it has been assumed that the dispersion matrix of the error vector sequence $\{\mathbf{e}(t)\}$ is known. In practice, they are unknown and needed to be estimated which can be done using the LS method. The LS method provides consistent estimators of the error variances and covariances. Also, given ω, using GLS method, the linear parameters are estimated as

$$\left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{M}_n(\omega)\right) \boldsymbol{\beta}^T = \left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{I}_2\right) \boldsymbol{W}_n^m(\omega),$$

where $\boldsymbol{\beta}$ is the parameter vector of order $2m$ and $\boldsymbol{W}_n^m(\omega)$ is a vector of order $2m$ with $(2j-1)$-th element is $\frac{1}{n}\sum_{t=1}^{n} y_j(t)\cos(\omega t)$ and $2j$-th element is $\frac{1}{n}\sum_{t=1}^{n} y_j(t)\sin(\omega t)$ for $j = 1, \ldots, m$. The 2×2 matrix $M_n(\omega)$ is same as defined in Section 3. Using an argument similar to the one presented at the end of Section 3, we observe that the GLSEs of the linear parameters as a function of ω are same as the LSEs as a function of ω.

6. Conclusions

In this article, we address the problem of estimation of parameters in a multichannel sinusoidal model. We have proposed two methods of estimation, the LSEs and GLSEs. If different sets of nearly periodic data are generated from the same system with same number of observations, then multichannel sinusoidal model is a viable option in analyzing such data sets simultaneously. As a result, the number of non-linear parameters, that is, the frequencies reduces to a great extent. The computational costs decreases substantially. We have also discussed both the LS and GLS estimation methods for a general m-channel model with single frequency and provided the theoretical properties of the estimators. It is observed that due to presence of a single nonlinear parameter, both the methods involve an one-dimensional optimization problem. Hence, the implementation of the proposed methods is quite simple in practice.

Acknowledgment

The authors thank the associate editor and the four unknown reviewers for their comments. Part of the work of the second author has been supported by a grant from the Science and Engineering Research Board, Government of India.

Data Availability Statement:

This manuscript has no associated data.

Appendix A

In this Appendix, we prove the strong consistency of the LSE, $\widehat{\boldsymbol{\xi}}^0$ of $\boldsymbol{\xi}^0$. We need the following lemmas to prove Theorem 2.1.

Lemma 1. *Let $\{e(t)\}$ be a sequence of i.i.d. random variables with mean zero and finite variance $\sigma^2 > 0$, then as $n \to \infty$,*

$$\sup_{\omega} \left| \frac{1}{n^{k+1}} \sum_{t=1}^{n} t^k e(t) \cos(\omega t) \right| \xrightarrow{a.s.} 0. \quad \text{and} \quad \sup_{\omega} \left| \frac{1}{n^{k+1}} \sum_{t=1}^{n} t^k e(t) \sin(\omega t) \right| \xrightarrow{a.s.} 0,$$

for $k = 0, 1, \ldots$

Lemma 2. *Let $\widetilde{\boldsymbol{\xi}} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{A}_2, \widetilde{B}_2, \widetilde{\omega})^T$ be an estimator of $\boldsymbol{\xi}^0$ that minimizes $R(\boldsymbol{\xi})$, defined in (6) and for any $\epsilon > 0$, let $S_\epsilon = \big\{\boldsymbol{\xi} : |\boldsymbol{\xi} - \boldsymbol{\xi}^0| > 5\epsilon\big\}$ for some fixed $\boldsymbol{\xi}^0 \in [-M, M] \times [-M, M] \times [-M, M] \times [-M, M] \times (0, \pi)$. If for any $\epsilon > 0$,*

$$\underline{\lim}_{n\to\infty} \inf_{S_\epsilon} \frac{1}{n} \big[R(\boldsymbol{\xi}) - R(\boldsymbol{\xi}^0)\big] \geq 0, \quad a.s. \tag{20}$$

then as $n \to \infty$, $\widetilde{\boldsymbol{\xi}} \to \boldsymbol{\xi}^0$ a.s.

Proof of Lemma 2: We write $\widetilde{\boldsymbol{\xi}}$ by $\widetilde{\boldsymbol{\xi}}_n$ and $R(\boldsymbol{\xi})$ by $R_n(\boldsymbol{\xi})$ to emphasize that these quantities depend on n. Suppose (20) is true but $\widetilde{\boldsymbol{\xi}}_n$ does not converges to $\boldsymbol{\xi}^0$ as $n \to \infty$. Then, there exists an $\epsilon > 0$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $|\widetilde{\boldsymbol{\xi}}_{n_k} - \boldsymbol{\xi}^0| > \epsilon$ for $k = 1, 2, \ldots$.

Therefore, $\widetilde{\boldsymbol{\xi}}_{n_k} \in S_\epsilon$ for all $k = 1, 2, \ldots$. By definition, $\widetilde{\boldsymbol{\xi}}_{n_k}$ is the estimator of $\boldsymbol{\xi}^0$ that minimizes $R_{n_k}(\boldsymbol{\xi})$ when $n = n_k$. This implies that

$$R_{n_k}(\widetilde{\boldsymbol{\xi}}_{n_k}) \le R_{n_k}(\boldsymbol{\xi}^0) \Rightarrow \frac{1}{n_k}\Big[R_{n_k}(\widetilde{\boldsymbol{\xi}}_{n_k}) - R_{n_k}(\boldsymbol{\xi}^0)\Big] \le 0.$$

Therefore, $\underline{\lim}_{n\to\infty} \inf_{\boldsymbol{\theta}_{n_k}\in S_\epsilon} \frac{1}{n_k}\left[R_{n_k}(\widetilde{\boldsymbol{\xi}}_{n_k}) - R_{n_k}(\boldsymbol{\xi}^0)\right] \le 0$. This contradicts inequality (20) and so, the result follows. ∎

Lemma 3. *If $\omega \in (0, \pi)$, then the following results hold.*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{2m+1}{2}}} \sum_{t=1}^{n} t^m \cos(\omega t) = \lim_{n\to\infty} \frac{1}{n^{\frac{2m+1}{2}}} \sum_{t=1}^{n} t^m \sin(\omega t) = 0, \quad m = 0, 1, 2.$$

Proof of Theorem 2.1: Write

$$\frac{1}{n}\Big[R(\boldsymbol{\xi}) - R(\boldsymbol{\xi}^0)\Big] = f(\boldsymbol{\xi}) + g(\boldsymbol{\xi})$$

where

$$\begin{aligned}
f(\boldsymbol{\xi}) &= \frac{1}{n}\sum_{t=1}^{n}\Big[A_1^0\cos(\omega^0 t) + B_1^0\sin(\omega^0 t) - A_1\cos(\omega t) - B_1\sin(\omega t)\Big]^2 \\
&\quad + \frac{1}{n}\sum_{t=1}^{n}\Big[A_2^0\cos(\omega^0 t) + B_2^0\sin(\omega^0 t) - A_2\cos(\omega t) - B_2\sin(\omega t)\Big]^2, \\
g(\boldsymbol{\xi}) &= \frac{2}{n}\sum_{t=1}^{n} e_1(t)\Big[A_1^0\cos(\omega^0 t) + B_1^0\sin(\omega^0 t) - A_1\cos(\omega t) - B_1\sin(\omega t)\Big] \\
&\quad + \frac{2}{n}\sum_{t=1}^{n} e_2(t)\Big[A_2^0\cos(\omega^0 t) + B_2^0\sin(\omega^0 t) - A_2\cos(\omega t) - B_2\sin(\omega t)\Big].
\end{aligned}$$

As $\{e_1(t)\}$ and $\{e_2(t)\}$ are sequences of i.i.d. random variables with mean zeros are finite variances, using Lemma 1, we have $\lim_{n\to\infty} \sup_{\boldsymbol{\xi}\in S_\epsilon} g(\boldsymbol{\xi}) = 0$ a.s. Now consider the sets

$$\begin{aligned}
S_{\epsilon,A_i} &= \Big\{\boldsymbol{\xi} : |A_i - A_i^0| > \epsilon\Big\}, \quad S_{\epsilon,B_i} = \Big\{\boldsymbol{\xi} : |B_i - B_i^0| > \epsilon\Big\}, i = 1, 2 \\
\text{and } S_{\epsilon,\omega} &= \Big\{\boldsymbol{\xi} : |\omega - \omega^0| > \epsilon\Big\}.
\end{aligned}$$

Then $S_\epsilon \subset S_{\epsilon,A_1} \cup S_{\epsilon,A_2} \cup S_{\epsilon,B_1} \cup S_{\epsilon,B_2} \cup S_{\epsilon,\omega}$. Therefore, to show

$$\underline{\lim} \inf_{\boldsymbol{\xi}\in S_\epsilon} f(\boldsymbol{\xi}) > 0 \quad \text{a.s.} \tag{21}$$

it is enough to show (21) over each of these sets $S_{\epsilon,A_1}, S_{\epsilon,A_2}, S_{\epsilon,B_1}, S_{\epsilon,B_2}$ and $S_{\epsilon,\omega}$. Now for any $\epsilon > 0$, taking infimum over the set S_{ϵ,A_1}, we have

$$\begin{aligned}
\underline{\lim} \inf_{\boldsymbol{\xi} \in S_{\epsilon,A_1}} f(\boldsymbol{\xi}) &= \underline{\lim} \inf_{\boldsymbol{\xi} \in S_{\epsilon,A_1}} \frac{1}{n} \sum_{t=1}^{n} \Bigg[\Big\{ A_1^0 \cos(\omega^0 t) + B_1^0 \sin(\omega^0 t) - A_1 \cos(\omega t) - B_1 \sin(\omega t) \Big\}^2 \\
&\quad + \Big\{ A_2^0 \cos(\omega^0 t) + B_2^0 \sin(\omega^0 t) - A_2 \cos(\omega t) - B_2 \sin(\omega t) \Big\}^2 \Bigg] \\
&= \underline{\lim} \inf_{\boldsymbol{\xi} \in S_{\epsilon,A_1}} \frac{1}{n} \sum_{t=1}^{n} \Big\{ A_1^0 \cos(\omega^0 t) + B_1^0 \sin(\omega^0 t) - A_1 \cos(\omega t) - B_1 \sin(\omega t) \Big\}^2 \\
&= \underline{\lim} \inf_{|A_1 - A_1^0| > \epsilon} \frac{1}{n} \sum_{t=1}^{n} \Big\{ A_1^0 \cos(\omega^0 t) - A_1 \cos(\omega t) \Big\}^2 \\
&= \inf_{|A_1 - A_1^0| > \epsilon} \frac{1}{2} (A_1 - A_1^0)^2 > \frac{1}{2} \epsilon^2 > 0 \quad \text{a.s.}
\end{aligned}$$

Similarly, it can be proved for $S_{\epsilon,A_2}, S_{\epsilon,B_1}, S_{\epsilon,B_2}$ and $S_{\epsilon,\omega}$. Therefore, we have

$$\underline{\lim} \inf_{\boldsymbol{\xi} \in S_{\epsilon,A_1}} f(\boldsymbol{\xi}) > 0 \quad \text{a.s.}$$

and the theorem follows. ∎

Appendix B

In this section, we obtain the asymptotic distribution of the estimators which minimizes $R(\boldsymbol{\xi})$ for single component two-channel model. This has been stated in Theorem 2.2. In order to obtain the joint asymptotic distribution of the LSE of $\widehat{\boldsymbol{\xi}}$, let $R'(\boldsymbol{\xi})$ and $R''(\boldsymbol{\xi})$ be the vector of first derivatives of order 5×1 and the matrix of second derivatives of order 5×5 of $R(\boldsymbol{\xi})$, respectively. Expanding $R'(\boldsymbol{\xi})$ around $\boldsymbol{\xi}^0$ using multivariate Taylor series expansion

$$R'(\widetilde{\boldsymbol{\xi}}) - R'(\boldsymbol{\xi}^0) = R''(\bar{\boldsymbol{\xi}})(\widetilde{\boldsymbol{\xi}} - \boldsymbol{\xi}^0), \tag{22}$$

where $\bar{\boldsymbol{\xi}}$ is a point on the line joining $\widetilde{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}^0$. Define a diagonal matrix $\mathbf{D_1}$ as

$$\mathbf{D_1} = \text{diag}\{n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{3}{2}}\}.$$

We note that $R'(\widetilde{\boldsymbol{\xi}}) = 0$ and (29) can be written as

$$\mathbf{D_1}^{-1}(\widetilde{\boldsymbol{\xi}} - \boldsymbol{\xi}^0) = [\mathbf{D_1} R''(\bar{\boldsymbol{\xi}}) \mathbf{D_1}]^{-1} [\mathbf{D_1} R'(\boldsymbol{\xi}^0)] \tag{23}$$

if $[\mathbf{D_1} R''(\bar{\boldsymbol{\xi}})\mathbf{D_1}]$ is an invertible matrix for large n. The elements of $R''(\boldsymbol{\xi})$ are all continuous function and using the consistency results of $\boldsymbol{\xi}$, we have

$$\lim_{n\to\infty}[\mathbf{D_1} R''(\bar{\boldsymbol{\xi}})\mathbf{D_1}] = \lim_{n\to\infty}[\mathbf{D_1} R''(\boldsymbol{\xi}^0)\mathbf{D_1}] = \boldsymbol{\Gamma}. \tag{24}$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{B_1}{2} \\ 0 & 1 & 0 & 0 & -\frac{A_1}{2} \\ 0 & 0 & 1 & 0 & \frac{B_2}{2} \\ 0 & 0 & 0 & 1 & -\frac{A_2}{2} \\ \frac{B_1}{2} & -\frac{A_1}{2} & \frac{B_2}{2} & -\frac{A_2}{2} & \frac{1}{3}(A_1^2 + B_1^2 + A_2^2 + B_2^2) \end{pmatrix} \tag{25}$$

Using the central limit theorem (Fuller(1996)), it follows that the normalized first derivative vector is asymptotically normally distributed

$$\mathbf{D_1} R'(\boldsymbol{\xi}^0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}) \tag{26}$$

where the symmetric matrix $\boldsymbol{\Sigma} = ((\Sigma_{jk}))$ is the asymptotic dispersion matrix of $R'(\boldsymbol{\xi}^0)\mathbf{D_1}$. The elements of $\boldsymbol{\Sigma}$ are given by

$$\begin{aligned}
&\Sigma_{11} = 2\sigma_1^2, \quad \Sigma_{12} = 0, \quad \Sigma_{13} = 2\sigma_{12}, \quad \Sigma_{14} = 0, \quad \Sigma_{15} = B_1\sigma_1^2 + B_2\sigma_{12}, \\
&\Sigma_{22} = 2\sigma_1^2, \quad \Sigma_{23} = 0, \quad \Sigma_{24} = 2\sigma_{12}, \quad \Sigma_{25} = -A_1\sigma_1^2 - A_2\sigma_{12}, \\
&\Sigma_{33} = 2\sigma_2^2, \quad \Sigma_{34} = 0, \quad \Sigma_{35} = B_2\sigma_2^2 + B_1\sigma_{12}, \\
&\Sigma_{44} = 2\sigma_2^2, \quad \Sigma_{45} = -A_2\sigma_2^2 - A_1\sigma_{12}, \\
&\Sigma_{55} = \frac{2}{3}\Big[\sigma_1^2(A_1^2 + B_1^2) + \sigma_2^2(A_2^2 + B_2^2) + 2\sigma_{12}(A_1A_2 + B_1B_2)\Big],
\end{aligned}$$

and this proves the Theorem.

Appendix C

In this Appendix, we prove the strong consistency of $\widetilde{\sigma}_j^2$ and $\widetilde{\sigma}_{12}$ for two-channel multiple sinusoidal model. If for $k = 1, \ldots, p$, $j = 1, 2$, $\widetilde{A}_{jk}$, $\widetilde{B}_{jk}$ and $\widetilde{\omega}_k$ are LSEs of A_{jk}^0, B_{jk}^0 and ω_k^0, respectively, then the LSE of σ_j^2 will be

$$\widetilde{\sigma}_j^2 = \frac{1}{n}\sum_{t=1}^{n}\Big[y_j(t) - \sum_{k=1}^{n}\Big(\widetilde{A}_{jk}\cos(\widetilde{\omega}_k t) + \widetilde{B}_{jk}\sin(\widetilde{\omega}_j t)\Big)\Big]^2.$$

We need the following lemma to prove the strong consistency of $\widetilde{\sigma}_j^2$ along with Lemma 1.

Lemma 4. *If $\widetilde{\omega}_k$ is the LSE of ω_k^0 for model (1), then as $n \to \infty$,*

$$n(\widetilde{\omega}_k - \omega_k^0) \to 0, \quad a.s.$$

Proof of Lemma 4: Observe that from equation (29), we have

$$R'(\widetilde{\boldsymbol{\xi}}) - R'(\boldsymbol{\xi}^0) = R''(\bar{\boldsymbol{\xi}})(\widetilde{\boldsymbol{\xi}} - \boldsymbol{\xi}^0). \tag{27}$$

Define a diagonal matrix $\mathbf{U} = \text{diag}\{1, 1, 1, 1, \frac{1}{n}\}$, then from equation (27), we can write

$$\mathbf{U}^{-1}(\widetilde{\boldsymbol{\xi}} - \boldsymbol{\xi}^0) = \Big[\frac{1}{\sqrt{n}}\mathbf{U}R''(\bar{\boldsymbol{\xi}})\mathbf{U}\frac{1}{\sqrt{n}}\Big]^{-1}\Big[\frac{1}{n}\mathbf{U}R'(\boldsymbol{\xi}^0)\Big].$$

Then, $\frac{1}{n}\mathbf{U} = \mathbf{D}_1$, therefore $\Big[\frac{1}{\sqrt{n}}\mathbf{U}R''(\bar{\boldsymbol{\xi}})\mathbf{U}\frac{1}{\sqrt{n}}\Big]^{-1} \longrightarrow \boldsymbol{\Gamma}^{-1}$ as $n \to \infty$ and because of Lemma 1, we have

$$\frac{1}{n}\mathbf{U}R'(\boldsymbol{\xi}^0) \longrightarrow 0, \quad a.s.$$

That proves the lemma. ∎

Proof of Theorem 2.3: Observe that for $j = 1, 2$,

$$\begin{aligned}
\widetilde{\sigma}_j &= \frac{1}{n}\sum_{t=1}^{n}\Big[y_j(t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big]^2 \\
&= \frac{1}{n}\sum_{t=1}^{n}e_j^2(t) + \frac{2}{n}\sum_{t=1}^{n}e_j(t)\Big[A_j^0\cos(\omega^0 t) + B_j^0\sin(\omega^0 t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big] \\
&\quad + \frac{1}{n}\sum_{t=1}^{n}\Big[A_j^0\cos(\omega^0 t) + B_j^0\sin(\omega^0 t) - \widetilde{A}_j\cos(\widetilde{\omega}t) - \widetilde{B}_j\sin(\widetilde{\omega}t)\Big]^2 \\
&= R_1(j) + R_2(j) + R_3(j) \quad \text{(say)}.
\end{aligned}$$

The second term $R_2(j)$ converges to zero almost surely because of Lemma 1 and the last term $R_3(j)$ converges to zero due to Lemma 4 for $j = 1, 2$. Since $R_1(j) = \frac{1}{n}\sum_{t=1}^{n}e_j^2(t)$ converges almost surely to σ_j^2, $j = 1, 2$ because of strong law of large numbers, we have $\widetilde{\sigma}_j^2 \overset{a.s.}{\to} \sigma_j^2$.

Now consider the estimator of σ_{12}.

$$\begin{aligned}
\widetilde{\sigma}_{12} &= \frac{1}{n}\sum_{t=1}^{n}\Big[y_1(t) - \widetilde{A}_1\cos(\widetilde{\omega}t) - \widetilde{B}_1\sin(\widetilde{\omega}t)\Big]\Big[y_2(t) - \widetilde{A}_2\cos(\widetilde{\omega}t) - \widetilde{B}_2\sin(\widetilde{\omega}t)\Big] \\
&= \frac{1}{n}\sum_{t=1}^{n} e_1(t)e_2(t) + \frac{2}{n}\sum_{t=1}^{n} e_1(t)\Big[A_2^0\cos(\omega^0 t) + B_2^0\sin(\omega^0 t) - \widetilde{A}_2\cos(\widetilde{\omega}t) - \widetilde{B}_2\sin(\widetilde{\omega}t)\Big] \\
&\quad + \frac{2}{n}\sum_{t=1}^{n} e_2(t)\Big[A_1^0\cos(\omega^0 t) + B_1^0\sin(\omega^0 t) - \widetilde{A}_1\cos(\widetilde{\omega}t) - \widetilde{B}_1\sin(\widetilde{\omega}t)\Big] \\
&\quad \frac{1}{n}\sum_{t=1}^{n}\Big[A_1^0\cos(\omega^0 t) + B_1^0\sin(\omega^0 t) - \widetilde{A}_1\cos(\widetilde{\omega}t) - \widetilde{B}_1\sin(\widetilde{\omega}t)\Big]\times \\
&\qquad\qquad \Big[A_2^0\cos(\omega^0 t) + B_2^0\sin(\omega^0 t) - \widetilde{A}_2\cos(\widetilde{\omega}t) - \widetilde{B}_2\sin(\widetilde{\omega}t)\Big] \\
&= T_1 + T_2 + T_3 + T_4 \quad \text{(say)}.
\end{aligned}$$

Note that using Lemma 1, T_2 and T_3 converges to zero almost surely and using Lemma 4, T_4 also converges to zero almost surely. Now observe that the sequence $\{e_(t)e_2(t)\}$ is an i.i.d. sequence of random variables with mean σ_{12} and variance $\sigma_1^2\sigma_2^2 - \sigma_{12}^2$. Therefore, T_1 converges to σ_{12} almost surely. This proves the theorem.

Appendix D

In this Appendix, we prove Theorem 3.1 which states the strong consistency of GLSE of $\boldsymbol{\xi}$. The following lemma similar to Appendix A is required.

Lemma 5. *Let $\widehat{\boldsymbol{\xi}} = (\widehat{A}_1, \widehat{B}_1, \widehat{A}_2, \widehat{B}_2, \widehat{\omega})^T$ be an estimator of $\boldsymbol{\xi}^0$ that minimizes $S(\boldsymbol{\xi})$, defined in (10) and for any $\epsilon > 0$, let $G_\epsilon = \big\{\boldsymbol{\xi} : |\boldsymbol{\xi} - \boldsymbol{\xi}^0| > 5\epsilon\big\}$ for some fixed $\boldsymbol{\xi}^0 \in [-M, M] \times [-M, M] \times [-M, M] \times [-M, M] \times (0, \pi)$. If for any $\epsilon > 0$,*

$$\underline{\lim}_{n\to\infty} \inf_{G_\epsilon} \frac{1}{n}\big[S(\boldsymbol{\xi}) - S(\boldsymbol{\xi}^0)\big] \geq 0, \quad a.s. \tag{28}$$

then as $n \to \infty$, $\widehat{\boldsymbol{\xi}} \to \boldsymbol{\xi}^0$ a.s.

Proof of Lemma 2: This lemma can be proved along the same line as Lemma 2, so it is omitted.

Proof of Theorem 3.1: In this proof, we denote

$$\mu_k(\omega; t) = A_k\cos(\omega t) + B_k\sin(\omega t), \quad \mu_k^0(\omega^0; t) = A_k^0\cos(\omega^0 t) + B_k^0\sin(\omega^0 t).$$

Write

$$\frac{1}{n}\Big[S(\boldsymbol{\xi}) - S(\boldsymbol{\xi}^0)\Big] = h_1(\boldsymbol{\xi}) + h_2(\boldsymbol{\xi}) + h_3(\boldsymbol{\xi})$$

where

$$\begin{aligned}
h_1(\boldsymbol{\xi}) &= \frac{\sigma_2^2}{n}\sum_{t=1}^{n}\Big(\mu_1^0(\omega^0;t) - \mu_1(\omega;t)\Big)^2 + \frac{\sigma_1^2}{n}\sum_{t=1}^{n}\Big(\mu_2^0(\omega^0;t) - \mu_2(\omega;t)\Big)^2 \\
&\quad -\frac{2\sigma_{12}}{n}\sum_{t=1}^{n}\Big(\mu_1^0(\omega^0;t) - \mu_1(\omega;t)\Big)\Big(\mu_2^0(\omega^0;t) - \mu_2(\omega;t)\Big) \\
h_2(\boldsymbol{\xi}) &= \frac{2\sigma_2^2}{n}\sum_{t=1}^{n} e_1(t)\Big(\mu_1^0(\omega^0;t) - \mu_1(\omega;t)\Big) - \frac{2\sigma_{12}}{n}\sum_{t=1}^{n} e_1(t)\Big(\mu_2^0(\omega^0;t) - \mu_2(\omega;t)\Big) \\
h_3(\boldsymbol{\xi}) &= \frac{2\sigma_1^2}{n}\sum_{t=1}^{n} e_2(t)\Big(\mu_2^0(\omega^0;t) - \mu_2(\omega;t)\Big) - \frac{2\sigma_{12}}{n}\sum_{t=1}^{n} e_1(t)\Big(\mu_1^0(\omega^0;t) - \mu_1(\omega;t)\Big).
\end{aligned}$$

According to Assumption 1, $\{e_1(t)\}$ and $\{e_2(t)\}$ are sequences of i.i.d. random variables with mean zeros are finite variances. Therefore, using Lemma 1, we have

$$\lim_{n\to\infty}\sup_{\boldsymbol{\xi}\in G_\epsilon} h_2(\boldsymbol{\xi}) = 0, \quad \text{and} \quad \lim_{n\to\infty}\sup_{\boldsymbol{\xi}\in G_\epsilon} h_3(\boldsymbol{\xi}) = 0.$$

Note that the proof will be complete if we can show $\liminf \inf_{\boldsymbol{\xi}\in G_\epsilon} h_1(\boldsymbol{\xi}) > 0$ a.s. Since

$$G_\epsilon \subset G_{\epsilon,A_1} \cup G_{\epsilon,A_2} \cup G_{\epsilon,B_1} \cup G_{\epsilon,B_2} \cup G_{\epsilon,\omega},$$

where

$$\begin{aligned}
G_{\epsilon,A_i} &= \big\{\boldsymbol{\xi} : |A_i - A_i^0| > \epsilon\big\},\ \ G_{\epsilon,B_i} = \big\{\boldsymbol{\xi} : |B_i - B_i^0| > \epsilon\big\}, i = 1,2 \\
\text{and}\ \ G_{\epsilon,\omega} &= \big\{\boldsymbol{\xi} : |\omega - \omega^0| > \epsilon\big\}.
\end{aligned}$$

Now for any $\epsilon > 0$, for each of the above sets $\liminf \inf_{\boldsymbol{\xi}\in G} h_1(\boldsymbol{\xi} > 0$ a.s. where G can be G_{ϵ,A_i}, G_{ϵ,B_i}, $k = 1,2$ or $G_{\epsilon,\omega}$. Therefore, the result follows.

Appendix E

In this Appendix, we prove Theorem 3.2. As the model considered in Theorem 3.1 is the two channel model with one frequency, the number of unknown parameters is 5.

Let $S'(\boldsymbol{\xi})$ and $S''(\boldsymbol{\xi})$ be a 5×1 vector and 5×5 matrix of $S(\boldsymbol{\xi})$, where $S(\boldsymbol{\xi})$ is defined in (10). Expanding $S'(\boldsymbol{\xi})$ around $\boldsymbol{\xi}^0$ using multivariate Taylor series expansion

$$S'(\widetilde{\widetilde{\boldsymbol{\xi}}}) - S'(\boldsymbol{\xi}^0) = S''(\bar{\boldsymbol{\xi}})(\widetilde{\widetilde{\boldsymbol{\xi}}} - \boldsymbol{\xi}^0), \tag{29}$$

where $\bar{\boldsymbol{\xi}}$ is a point on the line joining $\widetilde{\widetilde{\boldsymbol{\xi}}}$ and $\boldsymbol{\xi}^0$. Define a diagonal matrix $\mathbf{D_1}$ as

$$\mathbf{D_1} = \mathrm{diag}\{n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{3}{2}}\}.$$

We note that $S'(\widetilde{\widetilde{\boldsymbol{\xi}}}) = 0$ and (29) can be written as

$$\mathbf{D_1}^{-1}(\widetilde{\widetilde{\boldsymbol{\xi}}} - \boldsymbol{\xi}^0) = [\mathbf{D_1} S''(\bar{\boldsymbol{\xi}})\mathbf{D_1}]^{-1}[\mathbf{D_1} S'(\boldsymbol{\xi}^0)] \tag{30}$$

if $[\mathbf{D_1} S''(\bar{\boldsymbol{\xi}})\mathbf{D_1}]$ is an invertible matrix for large n. The elements of $S''(\boldsymbol{\xi})$ are all continuous function and using the consistency results of $\widetilde{\widetilde{\boldsymbol{\xi}}}$, we have

$$\lim_{n\to\infty} [\mathbf{D_1} S''(\bar{\boldsymbol{\xi}})\mathbf{D_1}] = \lim_{n\to\infty} [\mathbf{D_1} S''(\boldsymbol{\xi}^0)\mathbf{D_1}] = \mathbf{\Gamma}_g, \tag{31}$$

where the matrix $\mathbf{\Gamma}_g$ is given in Theorem 3.2. The elements of $S''(\boldsymbol{\xi}^0)$ are listed in Appendix F. The first order derivatives of $S(\boldsymbol{\xi})$ at the true value $\boldsymbol{\xi}^0$ are as follows.

$$\begin{aligned}
\left.\frac{\partial S(\boldsymbol{\xi})}{\partial A_1}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} &= -2\sigma_2^2 \sum_{t=1}^{n} e_1(t)\cos(\omega^0 t) + 2\sigma_{12} \sum_{t=1}^{n} e_2(t)\cos(\omega^0 t) \\
\left.\frac{\partial S(\boldsymbol{\xi})}{\partial B_1}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} &= -2\sigma_2^2 \sum_{t=1}^{n} e_1(t)\sin(\omega^0 t) + 2\sigma_{12} \sum_{t=1}^{n} e_2(t)\sin(\omega^0 t) \\
\left.\frac{\partial S(\boldsymbol{\xi})}{\partial A_2}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} &= -2\sigma_1^2 \sum_{t=1}^{n} e_2(t)\cos(\omega^0 t) + 2\sigma_{12} \sum_{t=1}^{n} e_1(t)\cos(\omega^0 t) \\
\left.\frac{\partial S(\boldsymbol{\xi})}{\partial B_2}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} &= -2\sigma_1^2 \sum_{t=1}^{n} e_2(t)\sin(\omega^0 t) + 2\sigma_{12} \sum_{t=1}^{n} e_1(t)\sin(\omega^0 t) \\
\left.\frac{\partial S(\boldsymbol{\xi})}{\partial \omega}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} &= 2\sigma_2^2 \sum_{t=1}^{n} t e_1(t)\Big(A_1^0\sin(\omega^0 t) - B_1^0\cos(\omega^0 t)\Big) \\
&\quad +2\sigma_1^2 \sum_{t=1}^{n} t e_2(t)\Big(A_2^0\sin(\omega^0 t) - B_2^0\cos(\omega^0 t)\Big) \\
&\quad -2\sigma_{12} \sum_{t=1}^{n} t e_1(t)\Big(A_2^0\sin(\omega^0 t) - B_2^0\cos(\omega^0 t)\Big) \\
&\quad -2\sigma_{12} \sum_{t=1}^{n} t e_2(t)\Big(A_1^0\sin(\omega^0 t) - B_1^0\cos(\omega^0 t)\Big)
\end{aligned}$$

Appendix F

In this Appendix, we list the second order derivatives of $S(\boldsymbol{\xi})$ with respect the elements of $\boldsymbol{\xi}$.

$$
\begin{aligned}
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_1^2} &= 2\sigma_2^2 \sum_{t=1}^{n} \cos^2(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_1^2} = 2\sigma_2^2 \sum_{t=1}^{n} \sin^2(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_1 B_1} = 2\sigma_2^2 \sum_{t=1}^{n} \cos(\omega t)\sin(\omega t), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_2^2} &= 2\sigma_1^2 \sum_{t=1}^{n} \cos^2(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_2^2} = 2\sigma_1^2 \sum_{t=1}^{n} \sin^2(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_2 B_2} = 2\sigma_1^2 \sum_{t=1}^{n} \cos(\omega t)\sin(\omega t), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_1 A_2} &= -2\sigma_{12} \sum_{t=1}^{n} \cos^2(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_1 B_2} = -2\sigma_{12} \sum_{t=1}^{n} \cos(\omega t)\sin(\omega t), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_1 A_2} &= -2\sigma_{12} \sum_{t=1}^{n} \cos(\omega t)\sin(\omega t), \quad \frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_1 B_2} = -2\sigma_{12} \sum_{t=1}^{n} \sin^2(\omega t),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial \omega^2} &= 2\sigma_2^2 \sum_{t=1}^{n} t^2 \Big(A_1 \sin(\omega t) - B_1 \cos(\omega t)\Big)^2 + 2\sigma_1^2 \sum_{t=1}^{n} t^2 \Big(A_2 \sin(\omega t) - B_2 \cos(\omega t)\Big)^2 \\
&\quad + 2\sigma_2^2 \sum_{t=1}^{n} t^2 \Big(y_1(t) - A_1 \cos(\omega t) - B_1 \sin(\omega t)\Big)\Big(A_1 \cos(\omega t) + B_1 \sin(\omega t)\Big) \\
&\quad + 2\sigma_1^2 \sum_{t=1}^{n} t^2 \Big(y_2(t) - A_2 \cos(\omega t) - B_2 \sin(\omega t)\Big)\Big(A_2 \cos(\omega t) + B_2 \sin(\omega t)\Big) \\
&\quad - 4\sigma_{12} \sum_{t=1}^{n} t^2 \Big(A_1 \sin(\omega t) - B_1 \cos(\omega t)\Big)\Big(A_2 \sin(\omega t) - B_2 \cos(\omega t)\Big) \\
&\quad - 2\sigma_{12} \sum_{t=1}^{n} t^2 \Big(y_1(t) - A_1 \cos(\omega t) - B_1 \sin(\omega t)\Big)\Big(A_2 \sin(\omega t) - B_2 \cos(\omega t)\Big) \\
&\quad - 2\sigma_{12} \sum_{t=1}^{n} t^2 \Big(y_2(t) - A_2 \cos(\omega t) - B_2 \sin(\omega t)\Big)\Big(A_1 \sin(\omega t) - B_1 \cos(\omega t)\Big), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_1 \partial \omega} &= -2\sigma_2^2 \sum_{t=1}^{n} t \cos(\omega t)\Big(A_1 \sin(\omega t) - B_1 \cos(\omega t)\Big) \\
&\quad + 2\sigma_{12} \sum_{t=1}^{n} t \cos(\omega t)\Big(A_2 \sin(\omega t) - B_2 \cos(\omega t)\Big) \\
&\quad - 2\sigma_{12} \sum_{t=1}^{n} t \sin(\omega t)\Big(y_2(t) - A_2 \cos(\omega t) - B_2 \sin(\omega t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial A_2 \partial \omega} &= -2\sigma_1^2 \sum_{t=1}^{n} t\cos(\omega t)\Big(A_2\sin(\omega t) - B_2\cos(\omega t)\Big) \\
&+2\sigma_{12} \sum_{t=1}^{n} t\cos(\omega t)\Big(A_1\sin(\omega t) - B_1\cos(\omega t)\Big) \\
&-2\sigma_{12} \sum_{t=1}^{n} t\sin(\omega t)\Big(y_1(t) - A_1\cos(\omega t) - B_1\sin(\omega t)\Big), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_1 \partial \omega} &= -2\sigma_2^2 \sum_{t=1}^{n} t\sin(\omega t)\Big(A_1\sin(\omega t) - B_1\cos(\omega t)\Big) \\
&+2\sigma_{12} \sum_{t=1}^{n} t\sin(\omega t)\Big(A_2\sin(\omega t) - B_2\cos(\omega t)\Big) \\
&-2\sigma_{12} \sum_{t=1}^{n} t\cos(\omega t)\Big(y_2(t) - A_2\cos(\omega t) - B_2\sin(\omega t)\Big), \\
\frac{\partial^2 S(\boldsymbol{\xi})}{\partial B_2 \partial \omega} &= -2\sigma_1^2 \sum_{t=1}^{n} t\sin(\omega t)\Big(A_2\sin(\omega t) - B_2\cos(\omega t)\Big) \\
&+2\sigma_{12} \sum_{t=1}^{n} t\sin(\omega t)\Big(A_1\sin(\omega t) - B_1\cos(\omega t)\Big) \\
&-2\sigma_{12} \sum_{t=1}^{n} t\cos(\omega t)\Big(y_1(t) - A_1\cos(\omega t) - B_1\sin(\omega t)\Big).
\end{aligned}
$$

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