Real Analysis:

Basic Concepts

1. Norm and Distance

- Recall that $\Re^n$ is the set of all n-vectors $x = (x_1, x_2, ..., x_n)$, where each x_i is a real number for $i = 1, 2, ..., n.$
  - The (Euclidean) **norm** of a vector $x \in \Re^n$ is denoted by $\|x\|$ and defined by

$$\|x\| = \left(x_1^2 + x_2^2 + ... + x_n^2\right)^{\frac{1}{2}} .$$

- **Properties of Norm:**

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda| \, . \, \|x\|$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality);

(4) $|xy| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz Inequality).

- **Distance** (or **Metric**): For $x, y \in \Re^n$, the distance between x and y, denoted by $d(x, y)$, is

$$d(x, y) = \|x - y\|.$$

  – Note that $d(x, 0) = \|x\|$. So norm of x can be interpreted as the distance of x from the 0 vector.

- **Properties of Distance:**

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

2. Sequences and Limits

- **Sequence:** A *sequence of real numbers* is an assignment of a real number to each natural number.
  - The notation $\{x_n\}$ means the sequence whose n-th term is x_n.

- **Examples:**

(a) $\{1, 2, 3, 4, ...\}$,

(b) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$,

(c) $\left\{1, \frac{1}{2}, 4, \frac{1}{8}, 16, ...\right\}$,

(d) $\left\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, ...\right\}$,

(e) $\{-1, 1, -1, 1, -1, ...\}$,

(f) $\left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...\right\}$.

- **Limit of a Sequence:**

  Let $\{x_1, x_2, x_3, ...\}$ be a sequence of real numbers. A real number x is called the limit of the sequence $\{x_n\}$ if given any real number $\epsilon > 0$, there is a positive integer N such that $|x_n - x| < \epsilon$ whenever $n \geq N$.

  – If the sequence $\{x_n\}$ has a limit, we call the sequence *convergent*.

  – If x is a limit of the sequence $\{x_n\}$, we say that *the sequence converges* to x and write

$$\lim_{n\to\infty} x_n = x, \text{ or simply } x_n \to x.$$

- **Examples:** Here are three sequences which converge to 0:

(1) $\left\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, ...\right\}$,

(2) $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ...\right\}$,

(3) $\left\{1, \frac{3}{1}, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{3}{3}, \frac{1}{4}, ...\right\}$.

- **Limit Point (or Accumulation Point or Cluster Point):**

  If $\{x_n\}$ is a sequence of real numbers and x is a real number, we say x is a *limit point* (or *accumulation point* or *cluster point*) of the sequence if given any real number $\epsilon > 0$, there are infinitely many elements x_n of the sequence such that $|x_n - x| < \epsilon$.

  – A limit is a special case of a limit point.

  – A sequence can have a number of different limit points, but only one limit.

- **Theorem 1:**

  A sequence can have at most one limit.

  – Proof: To be discussed in class.

- Proofs of most theorems on sequences and their limits require the triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\|\,, \text{ for any } x, y \in \Re^n,$$

  or the subtraction variant of the triangle inequality:

$$\|x - y\| \geq |\|x\| - \|y\||\,, \text{ for any } x, y \in \Re^n.$$

  – For proofs refer to Theorems 10.5 and 10.6 (page 219) of the textbook.

- **Theorem 2:**

  Let $\{x_n\}$ and $\{y_n\}$ be sequences with limits x and y, respectively. Then the sequence $\{x_n + y_n\}$ converges to the limit $x + y$.

  – Proof: To be discussed in class.

- **Theorem 3:**

  Let $\{x_n\}$ and $\{y_n\}$ be sequences with limits x and y, respectively. Then the sequence $\{x_n y_n\}$ converges to the limit xy.

  – Proof: To be discussed in class.

- **Theorem 4:**

  Let $\{x_n\}$ be a convergent sequence with limit x and b be a real number.

  (a) *If $x_n \le b$ for all n, then $x \le b$*.

  (b) *If $x_n \ge b$ for all n, then $x \ge b$*.

  – Proof: To be discussed in class.

3. Sequences and Limits in $\Re^n$

- A sequence in $\Re^n$ is an assignment of a vector in $\Re^n$ to each natural number.

- **Open Ball:**
  Let $\bar{x} \in \Re^n$ and ϵ be a positive number. An open ball (with centre $\bar{x}$ and radius ϵ) in $\Re^n$ is

$$B_\epsilon(\bar{x}) = \{x \in \Re^n : \|x - \bar{x}\| < \epsilon\}.$$

  – Intuitively, a vector y in $\Re^n$ is close to $\bar{x}$ if y is in some $B_\epsilon(\bar{x})$ for a small but positive ϵ. The smaller ϵ is, the closer y is to $\bar{x}$.

- A sequence of vectors in $\Re^n$, $\{x_1, x_2, x_3, ...\}$, is said to **converge** to the vector x if given any real number $\epsilon > 0$, there is a positive integer N such that $\|x_n - x\| < \epsilon$ whenever $n \geq N$.

  – That is, $x_n \in B_\epsilon(x)$ for all $n \geq N$.

  – The vector x is called the **limit** of the sequence.

- **Theorem 5:**

  A sequence of vectors in $\Re^n$ converges if and only if all n sequences of its components converge in $\Re^1$.

  – Proof: To be discussed in class.

  – Theorem 5 enables us to apply the results on sequences in $\Re^1$ to sequences in $\Re^n$.

- **Theorem 6:**

  Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences of vectors in $\Re^n$ with limits x and y, respectively; and let $\{c_n\}$ be a convergent sequence of real numbers with limit c. Then the sequence $\{c_n x_n + y_n\}$ converges to the limit $cx + y$.

  – Proof: To be discussed in class.

- **Limit Point (or Accumulation Point or Cluster Point):**
  The vector x in $\Re^n$ is a *limit point* (or *accumulation point* or *cluster point*) of the sequence of vectors in $\Re^n$, $\{x_n\}$, if given any real number $\epsilon > 0$, there are infinitely many elements x_n of the sequence such that $\|x_n - x\| < \epsilon$.

  – The uniqueness of limits in $\Re^n$ follows directly from Theorems 1 and 5.

- **Subsequence:**

  Let $\{x_n\}$ be a sequence of vectors in $\Re^n$ and $\{n_r\}$ be a *strictly increasing* sequence of natural numbers. Then the sequence $\{x_{n_r}\}$ is called a *subsequence* of the sequence $\{x_n\}$.

- **Examples:** Consider the sequence $\left\{\frac{1}{1}, \frac{3}{1}, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{3}{3}, \frac{1}{4}, \frac{3}{4}, ...\right\}$.

#1. Explain whether each of the following sets is a subsequence of this sequence:

(a) $\left\{\frac{1}{1}, \frac{3}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$,

(b) $\left\{\frac{3}{1}, \frac{3}{2}, \frac{3}{3}\right\}$,

(c) $\left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, , ...\right\}$.

- If a sequence has a limit point, it may nonetheless have no limit.
  - However, if a sequence has a limit point, then there exists a subsequence that converges to this limit point.

4. Open Sets

- A set $S \subset \Re^n$ is *open* (in $\Re^n$) if for every $x \in S$, there exists an open ball (with centre x and radius $\epsilon > 0$) in $\Re^n$ which is completely contained in S.

  – That is, $x \in S \Rightarrow$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset S$.

  – If $x \in \Re^n$, an open set S containing x is called an **open neighbourhood** of x.

    - Thus, an open ball (with centre x and radius $\epsilon > 0$) in $\Re^n$ is an open neighbourhood of x.

  – The word "open" has a connotation of "no boundary": from any point in the set one can always move a little distance in *any* direction and still be in the set.

    - Consequently, open sets cannot contain their "boundary points".

#2. Example: Use the definition to show that the open interval $(1, 2) = \{x \in \Re : 1 < x < 2\}$ is an open set.

- In discussing the concept of an open set, it is important to specify the *space* in which we are considering the set.
  - For example, the set $\{x \in \Re : 0 < x < 1\}$ is open in $\Re$. But the set $\left\{(x_1, x_2) \in \Re^2 : 0 < x_1 < 1,\ x_2 = 0\right\}$ is *not open* in $\Re^2$, although, graphically, the two sets "look the same".

- **Theorem 7:**

  Open balls are open sets.

  - Proof: To be discussed in class.

- **Theorem 8:**

  (a) *Any union of open sets is open.*

  (b) *The finite intersection of open sets is open.*

  - Proof: To be discussed in class.

– Note that the intersection of an *infinite* number of open sets need not be open.

- For example, consider the open intervals $S_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ in $\Re$.

  Note that $\bigcap_{n=1}^{\infty} S_n = \{0\}$, *not* an open set.

- **Interior of a Set:**

  Let S be a subset of $\Re^n$. The *interior* of S, denoted by *int* S, is the union of all open sets contained in S.

  – By definition, the interior of a set can be considered as the largest open set which is contained in the given set.

  – The interior of an open set S is the set S itself.

#3. Prove that the interior of an open set S is the set S itself.

5. Closed Sets

- A set $S \subset \Re^n$ is *closed* (in $\Re^n$) if whenever $\{x_n\}$ is a sequence of points of S that is convergent in $\Re^n$, we have $\lim\limits_{n\to\infty} x_n \in S$.
  - Thus, if x is a given point and if there are points in a closed set S which are arbitrarily close to x, then x must be in S too.
    - Consequently, a closed set must contain all its "boundary points", just the opposite of the situation with open sets.

#4. Example: Prove that the set

$$S = \left\{(x_1, x_2) \in \Re^2 : x_1 \geq 0, x_2 \geq 0, \text{ and } x_1^2 + x_2^2 \leq 1\right\}$$

is a closed set in $\Re^2$.

- **Complement of a Set:** If $S \subset \Re^n$, the complement of S (in $\Re^n$) is denoted S^c, and defined by $S^c = \{x \in \Re^n : x \text{ is not in } S\}$.

- **Theorem 9:**

  A set $S \subset \Re^n$ *is closed* (*in* $\Re^n$) *if and only if its complement*, S^c, *is open in* $\Re^n$.

  – Proof: To be discussed in class.

- Using Theorem 9 together with Theorem 8, we obtain the next theorem simply by using set-theoretic complementation: $\left(\bigcup_i S_i\right)^c = \bigcap_i S_i^c$.

- **Theorem 10:**

  (a) *Any intersection of closed sets is closed.*

  (b) *The finite union of closed sets is closed.*

  – Proof: To be discussed in class.

– Just as arbitrary intersections of open sets need not be open, so too arbitrary unions of closed sets need not be closed.

- For example, consider the closed intervals

$$S_n = \left[-\frac{n}{n+1}, \frac{n}{n+1} \right] \text{ for } n \geq 1.$$

Note that $\bigcup_{n \geq 1} S_n = (-1, 1)$, an open interval.

- There are many sets which are *neither open nor closed* in $\Re^n$.

#5. Show that the half-open interval $(a, b]$ in $\Re$ is neither open nor closed in $\Re$.

#6. Show that a line minus a point in a plane is neither open nor closed in the plane.

- There are only two sets which are *both open and closed* in $\Re^n$: $\Re^n$ itself and the empty set.

- Consider an arbitrary set $A \subset \Re^n$ and an arbitrary point $x \in \Re^n$. Then one of the following three possibilities must hold:

(1) There is an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subset A$.

- These points $x \in \Re^n$ constitute the **interior** of A.

(2) There is an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subset A^c$.

- These points $x \in \Re^n$ constitute the **exterior** of A.

(3) For every $\epsilon > 0$, $B_\epsilon(x)$ contains points of both A and A^c.

- These points $x \in \Re^n$ constitute the **boundary** of A.

6. Compact Sets

• **Bounded Set:** A set $S \subset \Re^n$ is *bounded* if it is contained in some open ball in $\Re^n$.

#7. Show that the closed interval $[1, 3]$ is bounded in $\Re$.

#8. Show that $\Re_+$, the set of non-negative real numbers, is not bounded in $\Re$.

• **Alternative Characterizations of Bounded Sets:**

(A) Set $S \subset \Re^n$ is *bounded* if there exists a real number B such that $\|x\| \leq B, \forall\, x \in S$.

#9. Prove that characterization (A) is equivalent to the definition of a bounded set.

(B) A set $S \subset \Re^n$ is *bounded* if there exists a positive real number b such that whenever $x \in S$, $|x_i| \leq b$ for all $i = 1, 2, ..., n$.

#10. Prove that characterization (A) is equivalent to characterization (B).

$\Rightarrow$ All these three characterizations are alternative ways to look at boundedness.

#11. Show that $S = \{(x_1, x_2) \in \Re^2 : x_1 \geq 0, x_2 \geq 0$, and $px_1 + qx_2 \leq I, (\, p, q, I > 0)\}$ is bounded using all the three alternative characterizations.

- **Compact Set:** A set $S \subset \Re^n$ is *compact* if and only if it is both closed and bounded.
- Examples:

(1) $S_1 = \{(x_1, x_2) \in \Re^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ is a compact set.

(2) $S_2 = \{(x_1, x_2) \in \Re^2 : 0 \leq x_1 < 1, 0 \leq x_2 < 1\}$ is *not* a compact set as it is not closed.

(3) $S_3 = \{(x_1, x_2) \in \Re^2 : 0 \leq x_1 < 1, 0 \leq x_2\}$ is *not* a compact set as it is not bounded.

- An important feature of compact sets is that any sequence defined on a compact set must contain a subsequence that converges to a point in the set.
  - This important result is known as the **Bolzano-Weierstrass Theorem**.

- **Theorem 11 (Bolzano-Weierstrass Theorem):**

  Let C be a compact subset in $\Re^n$ and let $\{x_n\}$ be any sequence in C. Then $\{x_n\}$ has a convergent subsequence whose limit lies in C.

  - Proof: The proof for compact subsets in $\Re$ will be discussed in class.

7. Continuous Functions

- **Functions:**

  Let $A \subset \Re^n$. A *function*, f, from A to $\Re^m$ (written f: $A \rightarrow \Re^m$) is a rule which associates with each point in A a *unique* point in $\Re^m$.

  – A is called the *domain* of f.

  – The set $f(A) = \{y \in \Re^m : y = f(x) \text{ for some } x \in A\}$ is the *range* or *image* of f.

  – In the special case where $m = 1$, f is called a *real-valued* function.

- If f: $A \rightarrow \Re^m$ is a function, we can define $f^1(x)$ as the first component of the vector $f(x)$ for each $x \in A$.

  – Then f^1 is a function from A to $\Re$.

  – Similarly, $f^2, f^3, ..., f^m$ can be defined.

  – These real-valued functions are called the *component functions* of f.

- Conversely, if $g^1, g^2, ..., g^m$ are m real-valued functions on A, we can define

$$g(x) = \left(g^1(x), g^2(x), ..., g^m(x)\right), \text{ for each } x \in A.$$

  – Then g is a function from A to $\Re^m$.

- **Limit of a Function:**

  Let $A \subset \Re^n$, f: $A \to \Re^m$, and $x_0 \in A$. Then

$$\lim_{x \to x_0} f(x) = y$$

  means that whenever $\{x_n\}$ is sequence in A which converges to x_0, the sequence $\{f(x_n)\}$ in $\Re^m$ converges to y.

  Alternative Definition:

  Let $A \subset \Re^n$, f: $A \to \Re^m$, and $x_0 \in A$. Then

$$\lim_{x \to x_0} f(x) = y$$

  means that given any $\epsilon > 0$, there is a number $\delta > 0$, such that if $x \in A$, and $0 < \|x - x_0\| < \delta$, then $\|f(x) - y\| < \epsilon$.

- Example:

#12. Show by using both the definitions that for the real-valued function $f(x) = 2x$, $\lim_{x \to 1} f(x)$ exists.

- The concept of limit of a function is silent about the behaviour of the function at the value x_0.
  - For example, consider the function
$$f(x) = \begin{cases} 2x, & x \neq 1, \\ 3, & x = 1. \end{cases}$$
    Show that $\lim_{x \to 1} f(x)$ exists.

- Continuity is distinct from the concept of limit in that the behaviour of the function at x_0 is also relevant.

- **Continuity of a Function:**

  Let $A \subset \Re^n$, f: $A \to \Re^m$, and $x_0 \in A$. The function f is *continuous* at x_0 if whenever $\{x_n\}$ is sequence in A which converges to x_0, then the sequence $\{f(x_n)\}$ in $\Re^m$ converges to $f(x_0)$.

  Alternative Definition:

  Let $A \subset \Re^n$, f: $A \to \Re^m$, and $x_0 \in A$. The function f is *continuous* at x_0 if given any $\epsilon > 0$, there is a number $\delta > 0$, such that if $x \in A$, and $0 < \|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.

  – The function f is said to be *continuous* on A if it is continuous at every point $x \in A$.

#13. Prove that the two alternative definitions of continuity are equivalent.

- **Properties of Continuous Functions:**

  The sequential characterization of continuity is very helpful in proving that algebraic combinations of continuous functions are still continuous.

- **Theorem 12:**

(a) *Let* f *and* g *be functions from* $\Re^n$ *to* $\Re^m$*. Suppose* f *and* g *are continuous at* x*. Then* $f + g$, $f - g$, *and* $f \cdot g$ *are all continuous at* x*.*

(b) *Let* f *and* g *be functions from* $\Re^n$ *to* $\Re$*. Suppose* f *and* g *are continuous at* x*, and* $g(x) \neq 0$*. Then the quotient function* $\frac{f}{g}$ *is continuous at* x*.*

– Proof: To be discussed in class.

- **Theorem 13:**

Let $f = \left(f^1, f^2, ..., f^m\right)$ *be a function from* $\Re^n$ *to* $\Re^m$*. Then* f *is continuous at* x *if and only if each of its component functions* $f^i : \Re^n \to \Re$ *is continuous at* x*.*

– Proof: Homework!!

- **Theorem 14 (Weierstrass Theorem):**

Let C *be a compact subset of* $\Re^n$ *and* $f : C \to \Re$ *be continuous on* C*. Then there exists* x_m *and* x_M *in* C *such that* $f(x_m) \leq f(x) \leq f(x_M)$ *for all* $x \in C$*.*

– Proof: To be discussed in class.

References

- Must read the following two chapters from the textbook:
  - Chapter 12 (pages 253 – 272): Limits and Open Sets;
  - Chapter 13 (pages 273 – 299): Functions of Several Variables.
- This material is standard in many texts on Real Analysis. You might consult

1. Rudin, W., *Principles of Mathematical Analysis*, (chapters 2, 3, 4),
2. Rosenlicht, M., *Introduction to Analysis*, (chapters 3, 4).

- Some of the material is also covered in

3. Spivak, M., *Calculus on Manifolds*, (chapter 1).